Homework #9 — PHYS 603 — Spring 2008 Professor Victor Yakovenko
Deadline: **Thursday, April 24, 2008, in class** Office: 2115 Physics

Web page: `http://www2.physics.umd.edu/~yakovenk/teaching/`

Textbook: Silvio Salinas, *Introduction to Statistical Physics*
Springer, 2001, ISBN 0-387-95119-9

Do not forget to write your name and the homework number!

Each problem is worth 10 points.

Ch. 10 Free Bosons: Bose-Einstein Condensation; Photon Gas

1. **Problem 10.3. Bose gas with an internal degree of freedom.**

   Assume that the gas is in 3 dimensions. Write down a general equation for determining the Bose-Einstein condensation temperature T_0, but do not solve it in general case, because it does not have a simple solution. What is the value of the chemical potential μ at T_0?

   Determine T_0 in the limiting cases $\epsilon_1 \ll \hbar^2 n^{2/3}/m$ and $\epsilon_1 \gg \hbar^2 n^{2/3}/m$ (take the limits $\epsilon_1 \to 0$ and $\epsilon_1 \to \infty$ to answer this question).

   Is there Bose-Einstein condensation in this problem in 2 dimensions and 1 dimension? Explain why.

2. **Temperature dependences for parabolic and linear dispersion.**

   Assuming that the degeneracy is $\gamma = 1$, the general expressions for the number of particles, internal energy, and entropy per unit volume of a gas are

   $$n = \frac{N}{V} = \int d\varepsilon\, D(\varepsilon)\, f(\varepsilon), \quad u = \frac{U}{V} = \int d\varepsilon\, D(\varepsilon)\, \varepsilon\, f(\varepsilon), \quad s = \frac{S}{V} = \int d\varepsilon\, D(\varepsilon)\, s(\varepsilon), \tag{1}$$

   where $s(\varepsilon) = [f(\varepsilon)+1]\ln[f(\varepsilon)+1] - f(\varepsilon)\ln f(\varepsilon)$ and $f(\varepsilon) = [e^{\beta(\varepsilon-\mu)} - 1]^{-1}$ are the entropy function and the occupation number for a Bose gas.

   (a) **Parabolic dispersion.**

   Consider Bose particles with the parabolic dispersion $\varepsilon = \boldsymbol{p}^2/2m$ in 3 dimensions. How the density of states $D(\varepsilon)$ depends on ε? (To what power of ε is the function $D(\varepsilon)$ proportional?)

   Consider this gas at a temperature T below the Bose-Einstein condensation temperature T_0, where $\mu = 0$. From Eq. (1), determine the temperature dependences of the following quantities (determine to what power of temperature they are proportional): the number of particles in the excited states n_e, the internal energy of the system u, the pressure P, the internal energy per excited particle u/n_e, the entropy s, the specific heat at a constant volume c_V.

(b) Linear dispersion.

Consider Bose particles in 3 dimensions with the linear dispersion $\varepsilon = c|\boldsymbol{p}|$ and zero chemical potential, such as photons or phonons.

Answer the same questions as in Part (a), starting from the density of states.

3. Problem 10.5. The Stefan-Boltzmann law.

This is the only problem where I ask for numerical estimates.

For the numerical value of an integral in Part (b), see the Appendix to the next problem.

In Part (c), calculate the energy flux from the Sun per unit area of the Earth per unit time (assuming that the area is perpendicular to the flux). Do NOT calculate the pressure of this radiation. Instead, estimate the area on the Earth that receives the solar power equal to the total energy consumption of the whole humankind. The total energy consumption of the humankind is estimates as some 15 TW = 1.5×10^{13} W (source `http://www.newparadigmjournal.com/May2007/physicallimit.htm`).

4. UMD qualifier problem, August 1999: Bose-Einstein condensation in a harmonic trap.

Most textbook discussions of Bose-Einstein condensation employ the unrealistic assumption that the particles are confined by a square-well potential. A more realistic model of atomic traps places the particle in a harmonic potential of the form $U_{\rm ext} = \frac{1}{2}m\omega^2 r^2$, where m is the particle mass, r is the distance from the center of the trap, and $\varepsilon_0 = \hbar\omega$ is the oscillator energy. Assume for simplicity that the bosons have no internal degrees of freedom, either being spinless or selected by magnetic substate, and that mutual interaction can be neglected.

(a) Show that a continuous approximation to the density of states for a spinless particle confined by an isotropic three-dimensional harmonic potential takes the form

$$D(\varepsilon) = \frac{\varepsilon^2}{2\varepsilon_0^3}. \qquad (2)$$

VMY Hint: Introducing dimensionless coordinate $\tilde{r} = r\sqrt{m\omega/\hbar}$ and momentum $\tilde{p} = p/\sqrt{m\hbar\omega}$, show that the energy of a particle in a harmonic trap can be written as $\varepsilon = \varepsilon_0(\tilde{r}^2 + \tilde{p}^2)/2 = \varepsilon_0 R^2/2$, where R is the radius in the 6-dimensional phase space of $\tilde{r}$ and $\tilde{p}$. Density of states can be obtained as $D = dN/d\varepsilon$. The number of states dN is proportional to the volume of a spherical shell of thickness dR in the 6-dimensional phase space. (The area of a 5-dimensional sphere is given in Appendix below.) The energy increment is $d\varepsilon = \varepsilon_0\, R\, dR$.

(b) Derive a relationship between the critical temperature of Bose-Einstein condensation T_c and the particle number N_c.

(c) Using the fact that the mean kinetic and potential energies are equal for a harmonic potential, deduce the mean value $\langle r^2 \rangle$ in terms of T, m, and ω.

Using this result, estimate the average density of bosons in the excited states of the trap. [Hint: Use the formula $V \sim \langle r^2 \rangle^{3/2}$ to estimate the volume V occupied by the Bose gas.] How does the critical density $n_c = N_c/V$ depend on temperature?

(d) Under what conditions should your analysis be valid?

(e) Discuss qualitatively whether Bose-Einstein condensation would happen in a harmonic potential in 2 dimensions and in 1 dimension. Compare with Problem 1.

Appendix. Possibly useful information:

The area of a 5-dimensional sphere is $S_5 = \pi^3 R^5$.

$$\int_0^\infty dx \frac{x^n}{e^x - 1} = n!\,\zeta(n+1), \qquad \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \zeta(2,\,3,\,4,\,5) \approx \{1.64,\,1.20,\,1.08,\,1.04\}$$

$\zeta(s)$ is called the Riemann zeta-function.

Bose-Einstein condensation in 2D will be discussed in the Physics Colloquium by Jean Dalibard on Tuesday, April 22, 2008.