**Homework #8** — PHYS 603 — Spring 2008 Professor Victor Yakovenko
Deadline: **Thursday, April 17, 2008, in class** Office: 2115 Physics

Web page: `http://www2.physics.umd.edu/~yakovenk/teaching/`

Textbook: Silvio Salinas, *Introduction to Statistical Physics*
Springer, 2001, ISBN 0-387-95119-9

**Do not forget to write your name and the homework number!**

**Each problem is worth 10 points.**

# Ch. 9 The Ideal Fermi Gas

1. **Problem 9.1. Compressibility of the Fermi gas. [10 points]**

   The compressibility is defined in Eq. (3.43). Only derive an equation for the compressibility and do not calculate a numerical value for metallic sodium.

2. **Entropy and spin susceptibility of the Fermi gas at $T \ll \varepsilon_F$. [20 points]**

   (a) **Entropy of the Fermi gas.**

   As discussed in Problem 3 of Homework 7, the entropy of the Fermi gas is

   $$S = \gamma V \int d\varepsilon\, D(\varepsilon)\, s(\varepsilon), \qquad s(\varepsilon) = -f(\varepsilon)\ln f(\varepsilon) - [1-f(\varepsilon)]\ln[1-f(\varepsilon)], \quad (1)$$

   where $f(\varepsilon) = [e^{\beta(\varepsilon-\mu)}+1]^{-1}$ is the Fermi distribution function, $\gamma$ is the number of fermion species ($\gamma = 2$ for spin 1/2), and $D(\varepsilon)$ is the energy density of states.

   Sketch the function $s(\varepsilon)$ in the case $T \ll \varepsilon_F$. Indicate the position, the width, and the height of the peak in this function.

   Given the shape of the function $s(\varepsilon)$, argue that Eq. (1) can be approximated at $T \ll \varepsilon_F$ as

   $$S \approx \gamma V D(\varepsilon_F) \int d\varepsilon\, s(\varepsilon), \quad (2)$$

   where $D(\varepsilon_F)$ is the density of states at the Fermi energy.

   Substituting the formula for $s(\varepsilon)$ from Eq. (1) into Eq. (2) and integrating by parts over $\varepsilon$, show that the entropy at $T \ll \varepsilon_F$ is given by the following expression

   $$S \approx -\beta\gamma V D(\varepsilon_F) \int d\varepsilon\, (\varepsilon-\mu)^2 f'(\varepsilon) = \frac{2\pi^2}{3} V D(\varepsilon_F) T, \quad (3)$$

   where we used Eq. (9.40) and $\gamma = 2$.

   (b) **Heat capacity of the Fermi gas.**

   From Eq. (3), obtain the heat capacity of Fermi gas at $T \ll \varepsilon_F$

   $$C_V = T\left(\frac{\partial S}{\partial T}\right)_V \approx \frac{2\pi^2}{3} V D(\varepsilon_F) T. \quad (4)$$

Check that Eq. (4) agrees with Eq. (9.47). (The textbook defines $c_V$ as $C_V/N$.)
Let us calculate the heat capacity in a different way. From the general equation $S = -\sum_j G_j\,[f_j \ln f_j + (1-f_j)\ln(1-f_j)]$ show that a variation of the entropy is

$$\delta S = \sum_j G_j\,(\varepsilon_j - \mu)\,\delta f_j = \frac{\gamma V}{T}\int d\varepsilon\,(\varepsilon-\mu)\,D(\varepsilon)\,\delta f(\varepsilon). \tag{5}$$

From Eq. (5) with $\gamma = 2$, the heat capacity is

$$C_V = T\left(\frac{\delta S}{\delta T}\right)_V \approx 2V D(\varepsilon_F)\int d\varepsilon\,(\varepsilon-\mu)\,\frac{df(\varepsilon)}{dT}. \tag{6}$$

Show that Eq. (6) gives the same result as Eq. (4). When calculating the derivative in $T$, argue that temperature dependence of the chemical potential (9.45) can be ignored, because it gives a correction to $C_V$ of a higher order in $T/\varepsilon_F$.

**(c) Spin susceptibility of the Fermi gas.**

From Eq. (9.60), the magnetization of a Fermi gas with spin 1/2 is

$$M = \mu_B(N_+ - N_-) = \mu_B V\int d\varepsilon\,D(\varepsilon)\,[f(\varepsilon-\mu_B H) - f(\varepsilon+\mu_B H)]. \tag{7}$$

Then, the spin susceptibility at $H = 0$ is

$$\chi = \left.\frac{dM}{dH}\right|_{H=0} = -2\mu_B^2 V\int d\varepsilon\,D(\varepsilon)\,f'(\varepsilon) \approx 2\mu_B^2 V D(\varepsilon_F). \tag{8}$$

Prove the last step in Eq. (8) for $T \ll \varepsilon_F$ and check that the result agrees with Eq. (9.64).

**(d) The Sommerfeld ratio.**

Using Eqs. (4) and (8), show that the following ratio, called the Sommerfeld ratio, is equal to one for the Fermi gas:

$$\frac{\pi^2}{3}\frac{\chi T}{\mu_B^2 C_V} = 1. \tag{9}$$

Eq. (9) relates the two different observable quantities (the spin susceptibility and the heat capacity) and can be verified experimentally. Measurements of both the heat capacity (4) and the spin susceptibility (8) can be used to determine the density of states $D(\varepsilon_F)$, and Eq. (9) is a consistency condition.

**(e) Generality of the results.**

Eq. (4) shows that the heat capacity is proportional to temperature, Eq. (8) shows that the spin susceptibility is independent of temperature, and Eq. (9) shows that the Sommerfeld ratio is equal to 1. Do these results depend on the dimensionality of space $d$ and on the exact form of the dispersion relation, e.g. on the parameter $a$ for $\varepsilon = c|\boldsymbol{p}|^a$?

**(f) Density of states.**

Express the density of states at the Fermi energy $D(\varepsilon_F)$ in terms of the Fermi momentum $p_F$ and the Fermi velocity $v_F$ for the one-, two-, and three-dimensional spaces ($d = 1$, 2, 3). Do the expressions depend on the exact form of the electron dispersion relation, e.g. on the parameter $a$ for $\varepsilon = c|\boldsymbol{p}|^a$?

3. **Problem 8.8. Different space dimensions and different dispersions. [10 points]**

In Part (b), calculate the Fermi momentum $p_F$, the Fermi energy $\varepsilon_F$, and the internal energy per particle $U/N$ at zero temperature for a given density of particles $n = N/V$.

In Part (c), you may use the result of the Problem 2.

*One of the most popular current subjects in condensed-matter physics is graphene, which is one atomic layer of graphite. In this material, the space dimensionality is $d = 2$, and the electron dispersion is linear in momentum, $a = 1$.*