Homework #5 — PHYS 603 — Spring 2008 Professor Victor Yakovenko
Deadline: **Thursday, March 27, 2008, in class** Office: 2115 Physics

Web page: http://www2.physics.umd.edu/˜yakovenk/teaching/

Textbook: Silvio Salinas, *Introduction to Statistical Physics*
Springer, 2001, ISBN 0-387-95119-9

Do not forget to write your name and the homework number!

Each problem is worth 10 points.

Ch. 5 Canonical Ensemble

1. **Problem 5.1, Spin chain with energy S_j^2 in a magnetic field.**

   Do all parts of this problem for a non-zero magnetic field $H \neq 0$! Consider two cases, $\mu_0 H < D$ and $\mu_0 H > D$. Make sure that your answers in this Problem agree with your answers in Problem 4.1 of Homework 4 in the limit $H \to 0$.

2. **Problem 5.4, Harmonic oscillators with odd n.**

   The energy spectrum discussed in this Problem would arise when the oscillators are restricted by an impenetrable barrier at $x = 0$, so that $x > 0$ for each oscillator. In this case, the quantum mechanics demands that the wave functions must vanish at $x = 0$. This condition is consistent only with the antisymmetric wave functions of the oscillator. Thus, the permitted quantum numbers are the odd numbers $n = 1, 3, 5, \ldots$ with the energies $\epsilon_n = \hbar\omega_0(n + 1/2)$, where $n = 0$ corresponds to the ground state.

3. **Problem 5.6, Partition function for N classical harmonic oscillators.**

4. **Modified Problem 5.7, Partition function for N classical harmonic oscillators from canonical perspective.**

   Calculate the partition function $Z(\beta)$ of N classical harmonic oscillators using the formula

   $$Z(\beta) = \int_0^\infty \Omega(E)\, e^{-\beta E}\, dE, \qquad (1)$$

   where E is the total energy of N oscillators, and $\Omega(E)$ is the density of states calculated in Problem 2.5. Find the maximum of the integrand in Eq. (1) and integrate in a small vicinity of the maximum for $N \gg 1$. Show that Eq. (1) gives the same result as Problem 5.6.

   From the mathematical point of view, $Z(\beta)$ is the Laplace transform of $\Omega(E)$ in Eq. (1). The original Problem 5.7 in the textbook asks to perform the inverse Laplace transform and obtain $\Omega(E)$ from $Z(\beta)$. This is more complicated than the direct Laplace transform, so I don't ask you to do that.

5. Entropy of the spin 1/2 paramagnet.

Let us consider an ideal paramagnet of spin 1/2 discussed in Ch. 4.3 and 5.1. To simplify equations, let us set the magnetic moment to unity: $\mu_0 = 1$. It can be easily restored in the final equations by replacing $H \to \mu_0 H$.

(a) Let us introduce the total number of particles N, the total magnetization M, and the magnetization per particle m:

$$N = N_\uparrow + N_\downarrow, \qquad M = N_\uparrow - N_\downarrow, \qquad m = M/N. \tag{2}$$

From the combinatorial formula $\Omega = N!/N_\uparrow!N_\downarrow!$, express the entropy per particle $s = S/N$ in terms of m.

Notice that $s(m)$ depends only on m and does not explicitly depend on H and T. Show that Eqs. (5.35) and (5.36) give the same formula for $s(m)$.

Sketch s vs. $-1 \le m \le 1$ and vs. $-\infty \le H/T \le +\infty$. Obtain s at $m = 0, \pm 1$ and at $H/T = 0, \pm\infty$ and explain physical meaning of the obtained values.

(b) The energy of the system per particle is $u = -mH$, so $du = -m\,dH - H\,dm$. Argue that the first term $dw = -m\,dH$ is the work done by the external magnetic field H, whereas the second term represents the entropy change $-H\,dm = T\,ds$. So, the energy equation can be written as

$$u = -mH, \quad du = -m\,dH - H\,dm = dw + T\,ds, \quad T\,ds = -H\,dm, \quad dw = -m\,dH. \tag{3}$$

Indeed, the term $H\,dm$ involves changing the occupation numbers of the spin states, which changes the entropy $s(m)$, without changing the external field. On the other hand, the term $m\,dH$ involves changing the external field without changing the magnetization m and the entropy $s(m)$. For an adiabatic process, we would have $du = -m\,dH$, because m is constant.

(c) Inspired by the UMD qualifier problem, January 2006: Adiabatic demagnetization of a paramagnetic salt.

Suppose we know (say, experimentally) that the magnetic susceptibility of a substance follow the Curie law:

$$\chi(T) = \left(\frac{\partial m}{\partial H}\right)_{T,H=0} = \frac{1}{T}. \tag{4}$$

Using Eqs. (3) and (4), calculate the dependence of the thermodynamic entropy $s(H,T)$ on H, up to some (unknown) function $s(0,T)$. *Hint:*

$$ds = -\frac{H}{T}\,dm = -\frac{H}{T}\left(\frac{\partial m}{\partial H}\right)_T dH = -\chi\,\frac{H}{T}\,dH.$$

By taking the integral, you can find $s(H,T)$ up to an unknown $s(0,T)$.

Compare your result with Eq. (5.35) in the limit $H \ll T$, where the Curie law is valid. Make sure that the results agree and find $s(0,T)$.

There is a typo in Eq. (5.39). The variable H should not be there.

(d) From $s(H,T)$ found in Part (c), obtain the specific heat c_H at a constant H.

6. Adiabatic demagnetization of a spin 1/2 paramagnet.
Inspired by the UMD qualifier problem, January 2006.

Adiabatic demagnetization of a spin 1/2 paramagnet is an important method for achieving ultralow temperatures in condensed matter systems, currently up to 90 μK. You can read more about this method in the book by Tony Guénault, *Statistical Physics*, Ch. 3.1.3, pp. 30–34.

(a) Consider the following Carnot cycle for a paramagnetic material with spins 1/2.

I. At a constant temperature T_1, increase the magnetic field from H_1 to H_2.

II. Adiabatically decrease the magnetic field to some value H_2', so that the temperature reaches the value T_2.

III. At the constant temperature T_2, decrease the magnetic field from H_2' to a properly selected value H_1'.

IV. Adiabatically increase the magnetic field to the initial value H_1, so that the temperature reaches the initial value T_1.

Draw a sketch of this cycle vs. H on the horizontal axis and T on the vertical axis. *Hint: The stages of this cycle are represented by the straight lines in these coordinates.* How does T change with H in the adiabatic process? *Hint: The entropy remains constant. Use Eq. (5.35).*

(b) For each of the stages I, II, III, and IV, answer the following questions:

i. Calculate the energy change Δu. Does the energy of the system increase or decrease in this stage?

ii. Calculate the entropy change Δs. Does the entropy of the system increase or decrease in this stage? Is the heat transferred out of the paramagnet or to the paramagnet from the environment?

iii. Calculate the work w done by the external magnetic field. Is this work positive or negative? Does it contribute toward the increase or decrease of the energy of the system?

To answer these questions, use Eq. (3) and the appropriate equations from the textbook.

(c) The efficiency of refrigeration $\eta = q_2/w$ can be defined as the ratio of the amount of heat q_2 absorbed during the stage III and the total work w done by the external field during the cycle. Derive an expression for η in terms of T_1 and T_2. Show that the efficiency of refrigeration goes to zero in the limit $T_2 \to 0$, so the refrigerator cannot reach $T = 0$.

(d) Write down an explicit expression for the amount of heat q_2 absorbed during the stage III in terms of T_1, T_2, H_1, and H_2. In order to maximize q_2, how the values of H_1, and H_2 should be selected with respect to T_1?