Homework #4 — PHYS 603 — Spring 2008 Professor Victor Yakovenko
Deadline: **Thursday, March 6, 2008, in class** Office: 2115 Physics

Web page: http://www2.physics.umd.edu/˜yakovenk/teaching/

Textbook: Silvio Salinas, *Introduction to Statistical Physics*
Springer, 2001, ISBN 0-387-95119-9

Do not forget to write your name and the homework number!

Each problem is worth 10 points.

Ch. 4 Microcanonical Ensemble

1. **Problem 4.1, Spin chain with energy S_j^2.**

2. **Problem 4.4, Entropy of the Boltzmann gas.**

   Derive the general equations shown in this problem. In the last sentence, I believe the answer $-k_B T \ln T$ is wrong. To derive the entropy of the Boltzmann gas, use the following procedure.

   In the spirit of Eq. (4.51), in the continuous limit the occupation numbers become

$$\frac{N_j}{N} = P_j \quad \rightarrow \quad P(p) = \frac{e^{-p^2/2mT}}{Z_1}, \tag{1}$$

   where the normalizing factor Z_1 is given by a formula similar to Eq. (4.49), but written more accurately

$$Z_1 = \int \frac{d^3p\, d^3r}{(2\pi\hbar)^2}\, e^{-p^2/2mT} = V \left(\frac{\sqrt{2\pi m T}}{2\pi\hbar} \right)^3 \tag{2}$$

   The entropy per particle $s = -\sum_j P_j \ln P_j$ becomes

$$s = -\int \frac{d^3p\, d^3r}{(2\pi\hbar)^2}\, P(p) \ln P(p) = -\int \frac{d^3p\, d^3r}{(2\pi\hbar)^2}\, P(p) \left(-\frac{p^2}{2mT} - \ln Z_1 \right) = \ln Z_1 + \frac{1}{T} \left\langle \frac{p^2}{2m} \right\rangle$$

   Finish the calculation and obtain the answer that we obtained in class for the entropy of the ideal gas. In the spirit of Eq. (4.61), we also need to subtract the term $\ln N! \approx N \ln(N/e)$ from the entropy S. In the end, you should obtain an expression given by Eqs. (8.70) and (8.71), where $\gamma = 1$ and the fraction under the logarithm in Eq. (8.71) should be inverted.

3. **UMD qualifier problem, August 2000: A fiber with two types of segments.**

   Hint: The system is similar to those considered in Examples 2 and 4 in Ch. 4.3.

   Suppose that a very long straight fiber consists of N identical segments, each having two states. The state labeled α has energy ε_α and length l_α along the fiber, while the state β has energy ε_β and length l_β. The energy of each component is independent of the states of its neighbors. Thus, the total energy and length of the fiber for a particular arrangement of components is

$$E = N_\alpha \varepsilon_\alpha + N_\beta \varepsilon_\beta, \qquad L = N_\alpha l_\alpha + N_\beta l_\beta,$$

where N_α and N_β are the numbers of segments in the states α and β; $N = N_\alpha + N_\beta$. The fiber is held under a constant tension force τ.

(a) Recall that the Boltzmann entropy is $S = -k_B \sum_i P_i \ln P_i$, where the sum extends over all states occupied with probabilities $P_i = N_i/N$. Use the method of Lagrange multipliers to deduce the distribution of P_i that maximizes entropy subject to constraints that specify the ensemble averages $\langle E \rangle$ for energy and $\langle L \rangle$ for length. Identify the temperature and tension. This part is intended to be general, applicable to any one-dimensional system with constant temperature and tension.

(b) For this particular system, find an explicit expression for the dependence of length $\langle L \rangle$ upon temperature and tension. Sketch $\langle L \rangle$ as a function of the dimensionless parameter

$$\eta = \beta[(\varepsilon_\beta - \varepsilon_\alpha) - \tau(l_\beta - l_\alpha)].$$

Make the sketch for η from $-\infty$ to $+\infty$. Explain the physical meaning of $\eta = -\infty,\ 0,\ +\infty$.

(c) Demonstrate that at high temperatures one obtains a version of Hook's law by evaluating the isothermal elastic modulus

$$k = \left(\frac{\partial \tau}{\partial L}\right)_T .$$

Explain the temperature dependence of k.

Hint: Calculate $1/k = (\partial \langle L \rangle / \partial \tau)_T$ *using the expression for* $\langle L \rangle$ *vs.* τ *obtained in Part (b).*

(d) Suppose that the temperature is low enough so that $|\eta| \gg l_\beta/l_\alpha$. How does L depend upon τ? Describe the behavior of the fiber as τ is varied while the temperature is constant.

4. UMD qualifier problem, January 2000: Fluctuations in an *LC* circuit.

An interesting thermometer may be made by observing the voltage fluctuations in a circuit consisting of an ideal inductor L connected in parallel to an ideal capacitor C. The circuit is in thermal contact with a heat reservoir at temperature T.

(a) Write down an expression for the electric energy stored in the capacitor C and magnetic energy stored in the inductor L. Show that this system is a harmonic oscillator with appropriate analogs of mass and spring constant. What is the natural frequency ω_0 of this oscillator?

Hint: If you forgot the undergraduate E & M, here is the summary. The energy of the LC circuit is

$$E = \frac{Q^2}{2C} + \frac{LI^2}{2},$$

where Q is the electric charge on the capacitor, and I is the electric current through the inductor. The variables Q and $I = \dot{Q}$ are the conjugate variables analogous to the coordinate x and velocity $v = \dot{x}$ of an oscillator.

(b) Using classical statistical mechanics, deduce the root-mean-square voltage in the circuit $V_{\rm rms} = \sqrt{\langle V^2 \rangle}$.

Hint: The energy of the capacitor can be rewritten as $CV^2/2$, where $V = Q/C$ is voltage. As usual for an oscillator, the average energy of the capacitor is a half of the total average energy in the circuit: $C\langle V^2 \rangle/2 = \langle E \rangle/2$.

(c) Assuming that the oscillator energy states follow the quantum mechanical rule $E_n = (n + 1/2)\hbar\omega_0$, where ω_0 is the classical frequency of the oscillator, $\hbar$ is the Planck constant, and $n = 0, 1, 2, \ldots$, deduce the temperature dependence of $V_{\rm rms}$.

(d) Sketch $V_{\rm rms}$ vs. T. Evaluate the low and high T behavior of $V_{\rm rms}$ and compare with the classical result. For what range of T would this thermometer be useful?