Homework #11 — PHYS 603 — Spring 2008 Professor Victor Yakovenko
Deadline: **Thursday, May 8, 2008, in class** Office: 2115 Physics

Web page: `http://www2.physics.umd.edu/~yakovenk/teaching/`

Textbook: Silvio Salinas, *Introduction to Statistical Physics*
Springer, 2001, ISBN 0-387-95119-9

Do not forget to write your name and the homework number!

Each problem is worth 10 points.

Ch. 12 Phase Transitions. Ch. 13, The Ising Model, is also recommended. This is the last homework!

1. **Modified UMD qualifier problem, August 2002: Boiling and freezing of water at high elevation, the Clausius-Clapeyron relation.**

(a) The pressure in a liquid or gas decreases as we move away from the Earth. Show that, for a small vertical displacement dz, the change of pressure dP is

$$dP = -\rho\, g\, dz \tag{1}$$

where $\rho(z)$ is the mass density of the fluid.

(b) Assuming that the air near the Earth can be treated as a perfect gas in equilibrium with a constant temperature T, show that Eq. (1) yields the Boltzmann distribution for the variation of the number density $n(z)$ with the height z. (It is also called the barometric distribution.)

(c) Using the Clausius-Clapeyron relation (12.5)

$$\frac{dP}{dT} = \frac{s_G - s_L}{v_G - v_L} = \frac{q}{T\,\Delta v}, \tag{2}$$

prove that water (ice) boils (melts) at lower (higher) temperatures when going from the sea level to the top of a mountain.

In Eq. (2), $s_{L,G}$ is the entropy per molecule in the liquid or gas phase, $v_{L,G}$ is the volume per molecule in the liquid or gas phase, q is the latent heat per molecule released in the transition, and Δv is the volume change per molecule at the transition.

(d) At a given pressure P, the volume per molecule in the liquid phase is usually much smaller than in the gas phase, so it can be approximately neglected in Eq. (2): $\Delta v = v_G - v_L \approx v_G$. Assuming that q is constant and using the equation of state for the ideal gas $Pv_G = T$, integrate Eq. (2) and determine how the boiling temperature T depends on pressure P.

2. Expanded Problem 12.5. The Curie-Weiss and Landau theories of ferromagnetism.

Consider the Curie-Weiss model of ferromagnetism, Ch. 12.2. To answer the following questions, you can use the Curie-Weiss equation (12.47)

$$m = \tanh(\beta H + \beta\lambda m) \tag{3}$$

or the Landau expansion (12.74) of the Gibbs energy at T near T_c (where $T_c = \lambda$)

$$g(T, H; m) = f_0(T) - Hm + a(T - T_c)m^2 + bm^4. \tag{4}$$

Please derive the results for T near T_c using the Landau expansion (4). Keep in mind that $g(T, H; m)$ should be minimized with respect to m, i.e. $\partial g/\partial m = 0$.

(a) Obtain an asymptotic expression for the isothermal magnetic susceptibility $\chi = (\partial m/\partial H)_T$ at $T = T_c$ for small H as a function of H. Use Eq. (4) and verify that you result agrees with Eq. (12.58).

(b) Obtain asymptotic expressions for the spontaneous magnetization m_0 for $T \ll T_c$ (i.e. for $T \to 0$) and for T close to T_c (i.e. $(T - T_c) \to 0-$). Assume that $H = 0$. Use Eq. (3) for the first question and Eq. (4) for the second question. Compare with Eq. (12.51). Sketch the spontaneous magnetization m_0 vs. temperature T.

(c) Obtain asymptotic expressions for the isothermal magnetic susceptibility χ for T near T_c as functions of $T - T_c$ in the limit $H \to 0$.

Use Eq. (4) to obtain $m(T, H)$ for $T > T_c$ and $\chi = (\partial m/\partial H)_T$. For $T < T_c$, keep in mind that $m(T, H) = m_0(T) + \Delta m(T, H)$, where $m_0(T)$ is the spontaneous magnetization, and $\Delta m(T, H)$ is the additional magnetization induced by application of the infinitesimal magnetic field H. The magnetic susceptibility for $T < T_c$ is defined as $\chi = (\partial \Delta m/\partial H)_T$. Verify that your results agree with Eqs. (12.54) and (12.56).

(d) From Eq. (4) at $H = 0$, obtain the entropy $s = -(\partial g/\partial T)_H$ as a function of T for $T > T_c$ and $T < T_c$. Sketch $s(T)$ for T close to T_c. Is there a jump of s at T_c or is $s(T)$ a continuous function at T_c? Is there a cusp in $s(T)$ at T_c?

Extend your sketch of $s(T)$ to a wide range of temperatures from $T = 0$ to high T using your general knowledge of statistical physics. Assuming that we consider the spin 1/2, what is the entropy of unordered paramagnet at high $T > T_c$? What is the entropy of an ordered ferromagnet at $T = 0$?

(e) Using the results from the previous part, calculate the specific heat $c = T(\partial s/\partial T)_H$ at $H = 0$ for T close to T_c. Is there a jump of c when T crosses T_c? Is the jump up or down? Sketch $c(T)$ in a wide range of temperatures from $T = 0$ to high T. What is the value of c at $T = 0$ (use general reasoning – the Nernst theorem)?

3. Expanded Problem 12.3. Critical behavior of compressibility in the van der Waals model.

Solve this Problem using the Landau expansion for the Gibbs energy (12.27) with the coefficients given by Eqs. (12.22) – (12.25). Notice some similarity between Eq. (12.27) and Eq. (4). The variable $\psi = v - v_c$ plays the role of m, and the pressure $P - P_c$ plays the role of H. Determine the exponents γ and γ_p in the following relations.

(a) Calculate $\kappa_T(T, v = v_c) \propto (T - T_c)^{-\gamma}$ as a function of $T - T_c$ for $T > T_c$ at $P = P_c$. This calculation is analogous to the calculation of χ in Problem 2c. Compare with Eq. (12.54).

(b) Calculate $\kappa_T(T = T_c, P) \propto (P - P_c)^{-\gamma_p}$ as a function of $P - P_c$ for $P > P_c$ at $T = T_c$. This calculation is analogous to the calculation of χ in Problem 2a. Compare with Eq. (12.58).

4. Reformulated Problem 12.1. Coexistence curve in the van der Waals model.

Do not use the method described in the textbook formulation of the problem. Use the Landau expansion (12.30) with the coefficients (12.34) – (12.30) as discussed below.

Sketch a plot of the Gibbs energy g given by Eq. (12.30) as a function of ψ'. First, consider the case where $A_1 = 0$. How many minima $g(\psi')$ has for $t > 0$ and for $t < 0$? Are the depths of the two minima the same? How do the positions ψ_1 and ψ_2 of the minima depend of t for $t < 0$? Compare with Eq. (12.74) for the spontaneous magnetization m at $H = 0$.

How do these results change when $A_1 \neq 0$? How many minima $g(\psi')$ has for $t > 0$ and for $t < 0$? Are the depths of the minima the same?

For a given $t < 0$, find $\Delta p < 0$ such that $A_1 = 0$ in Eq. (12.34). What is the geometrical meaning of the pressure P that satisfies this conditions in Fig. 12.2?

Using these results, show that $v_G - v_L \propto \sqrt{P_c - P}$, where v_G and v_L are the boundaries of the straight horizontal line in Fig. 12.2. This means that the dashed boundary of the coexistence region in Fig. 12.2 is a parabola.

Similarly, using Eqs. (12.28), (12.29), and (12.23), show that $v_G + v_L \propto (P_c - P)$ in Fig. 12.2. This means that the parabola, which gives the dashed boundary of the coexistence region in Fig. 12.2, is not symmetric, but is tilted. Sketch how the parabola is tilted.