MIDTERM EXAM — PHYS 603 — Spring 2008 Professor Victor Yakovenko
Tuesday, March 11, 2008, 9:30–10:45 am Office: 2115 Physics

You may use the textbook, your notes, homework, and any other materials.

Do not forget to write your name!

Each problem is worth 10 points.

1. UMD qualifier problem, January 2004: Probability distribution of money.

This problem explores an analogy between the Boltzmann-Gibbs probability distribution of energy in statistical physics and the probability distribution of money in a closed system of economic agents. Consider a system consisting of $N \gg 1$ economic agents. At a given moment of time, each agent i has a non-negative amount of money $m_i \geq 0$ (debt is not permitted). As the agents engage in economic activity, money is constantly transferred between the agents in the form of payments. However, the total amount of money is conserved in binary transactions between agents: $m_i + m_j = m_i' + m_j'$. This condition is analogous to conservation of energy in collisions between atoms in a gas. We also assume that the system is closed, so the total amount of money in the system M is also conserved.

We wish to obtain a formula for the money distribution function $P(m)$ for the system in statistical equilibrium. It is defined so that $P(m)\,dm$ is the fraction of agents with money in the interval $[m, m+dm]$ and satisfies the standard normalization conditions:

$$\int_0^\infty P(m)\,dm = 1, \qquad \int_0^\infty m\,P(m)\,dm = \langle m \rangle = M/N, \tag{1}$$

where $\langle m \rangle = M/N$ is the average amount of money per agent.

(a) Let us divide the money semi-axis $m \geq 0$ into equal intervals Δm and count the number of agents belonging to each interval: N_1, N_2, $N_3, \ldots$ Obviously $\sum_{r=1}^{\infty} N_r = N$. This configuration can be realized in many different ways by moving agents between the intervals while preserving the occupation numbers N_1, N_2, $N_3, \ldots$ Write a combinatorial formula for the number of ways W a given distribution N_1, N_2, $N_3, \ldots$ can be realized.

(b) Using the Stirling approximate formula $\ln n! \approx n \ln n - n$ for $n \gg 1$, obtain an expression for $S = \ln W$, the entropy of the distribution.

(c) Using the method of Lagrange multipliers, find the configuration N_r^* that maximizes entropy S under constraints that the total number of agents is conserved and the total amount of money is conserved:

$$\sum_{r=1}^{\infty} N_r = N, \qquad \sum_{r=1}^{\infty} m_r\,N_r = M. \tag{2}$$

(d) Obtain the money distribution function $P(m)$ for the fraction of agents belonging to a given interval Δm: $P(m_r)\,\Delta m = N_r^*/N$. Determine the values of the Lagrange multipliers from the normalization conditions (1).

(e) Compare the obtained result for $P(m)$ with the Boltzmann-Gibbs formula for the probability distribution of energy $P(E)$ in physics. What is the analog of temperature in the system of economic agents? Compare with $\langle m \rangle$.

2. Modified Problem 4.5 from the textbook by Salinas.

Consider a lattice gas of N particles distributed among V cells (with $N \leq V$). Suppose that each cell may be either empty or occupied by a single particle. The number of microscopic states of this system will be given by

$$\Omega(V, N) = \frac{V!}{N!(V-N)!}.$$

(a) Obtain an expression for the entropy per particle, $s = S/N = s(v)$, where $v = V/N$.

(b) From this fundamental equation, obtain an expression for the equation of state p/T, where p is pressure and T is temperature. *Hint: Use Eq. (3.13).*

(c) Write an expansion of p/T in terms of the density $\rho = 1/v = N/V$ for small ρ. Show that the first term of this expansion gives the Boyle law of the ideal gases.

(d) Obtain a formula for μ/T, where μ is the chemical potential, in terms of v.
Hint: Use Eq. (3.10), or use Eq. (3.54) with $U = 0$ *and Eq. (3.58).*

(e) Sketch a graph of μ/T vs. v. What is the behavior of the chemical potential in the limits $v \to \infty$ and $v \to 1$? Keep in mind that $v \geq 1$ by construction.

3. Thermodynamic entropy for a gas of diatomic molecules.

The internal energy U and pressure p for an ideal gas consisting of diatomic molecules are given the following equations

$$U = \frac{5}{2}NT, \qquad pV = NT, \tag{3}$$

where N is the number of molecules, T is temperature, and V is volume. Here the contribution $(3/2)T$ comes from the 3 translational degrees of freedom, and the contribution $(2/2)T$ comes from 2 rotational degrees of freedom, as will be discussed in later chapters. We assume that the molecules are rigid objects, where relative vibrational motion of the atoms is not excited. Take Eq. (3) for granted; you don't need to derive or justify it.

(a) By integrating the equation $T\,dS = dU + p\,dV$, derive the entropy $S(T,V)$ as a function of T and V for a fixed number of molecules N, up to an (unknown) function of N.

By demanding that S is an extensive variable, obtained a well-behaved expression for the entropy per particle $s = S/N$, where s depends only on the intensive variables T and V/N.

(b) For an adiabatic expansion of this gas, derive relations between T and V, as well as between p and V. The entropy S is constant for an adiabatic expansion.