Homework #7 — PHYS 603 — Spring 2007
Deadline: **Tuesday, May 8, 2007, in class**

Professor Victor Yakovenko
Office: 2115 Physics

Web page: http://www2.physics.umd.edu/˜yakovenk/teaching/

Textbook: Gregory H. Wannier, *Statistical Physics*
Dover 1987 reprint of the 1966 edition, ISBN 0-486-65401-X

Do not forget to write your name and the homework number!

Each problem is worth 10 points.

Ch. 9 Quantum Statistics of Independent Particles

1. **Problem 9.1, Applying the three statistics to a two-level system.**

   Derive answers analytically. Do not substitute numbers.

2. **Problem 9.2, No Bose-Einstein condensation for free particle in** $D = 1, 2$.

3. **Problem 9.3, Adiabatic relation holds for any statistics.**

   You may use virial theorem (2.39).

4. **Problem 9.4, Entropy for Fermi statistics.**

5. **Problem 9.5, Entropy for any statistics.**

6. **Problem 9.6, Entropy for Bose statistics.**

   Also read Problem 9.7 and its solution for your information. You don't need to submit it, because solution is written in the book.

7. **UMD qualifier problem, January 2005: 1D Fermi gas.**
   Use results of Sec. 14-4 for this Problem.

   Consider a one-dimensional (1D) noninteracting free electron gas moving along the x axis with a number density of n electrons per unit length. Assume the electronic wavefunctions to be the usual plane wave momentum eigenstates with the electron energy being quadratic in momentum: $\varepsilon = p^2/2m$. Give the answers to the questions below in terms of Planck's constant $\hbar$, the electron mass m, Boltzmann's constant k_B, the 1D electron density n, and the absolute temperature T (as needed).

   (If you cannot derive an analytic form for an integral, give your answer in terms of the integral itself.)

   (a) Remembering that electron are fermions with spin 1/2, obtain an expression for the energy density of states $g(\varepsilon)$ of the 1D system.

   (b) Obtain an expression for the Fermi energy E_F of the system. (E_F is the $T = 0$ limit of the chemical potential μ.)

(c) Obtain an expression for the chemical potential of the system at an arbitrary temperature. Comment on the low and high temperature limits of the chemical potential.

(d) Obtain an integral expression for the specific heat of the system at an arbitrary temperature.

(e) Obtain the leading order temperature dependence of the specific heat in terms of an expansion in $k_B T/E_F$.

8. UMD qualifier problem, August 1998: Sticky surface and a gas.
Use results of Ch. 8 for this Problem.

The surface of a solid is in equilibrium with a surrounding ideal, monoatomic gas at temperature T in which the atoms have mass m. The solid has N_0 sites to which gas atoms can stick with at most one atom per site. Each site has zero energy when it is unoccupied and has energy $-\varepsilon_0$ when is occupied by an atom.

(a) Write down an expression for the grand partition function of the solid Z as an function of T and the chemical potential μ. Hint: You will need to know the number of ways in which N of the N_0 sites can be occupied. *VMY Hint: Compare the obtained partition function with that of fermions.*

(b) Let $x = \langle N\rangle/N_0$ where $\langle N\rangle$ is the average number of occupied sites. Use your result from Part 8a to obtain a relationship between x and μ. *VMY Hint: Compare with the average occupation number for fermions.*

(c) Show that the grand partition function for the gas in a volume V is given by

$$Z = \exp\left(\frac{V}{\lambda^3}e^{\beta\mu}\right), \qquad \beta = \frac{1}{k_B T}, \qquad \lambda = \sqrt{\frac{2\pi\hbar^2}{mk_B T}}. \tag{1}$$

(d) Use your result in Part 8c to relate pressure P of the ideal gas to the chemical potential μ. *VMY Hint: See Eq. (8.15).*
Comparing this with your result in Part 8b, show that

$$x = \frac{1}{1 + P_0/P} \tag{2}$$

and determine the dependence of P_0 on T and the other parameters of the problem.

9. UMD qualifier problem, August 1999: Bose-Einstein condensation in a harmonic trap. *Use results of Sec. 9.5 for this Problem.*

Most textbook discussions of Bose-Einstein condensation employ the unrealistic assumption that the particles are confined by a square-well potential. A more realistic model of atomic traps places the particle in a harmonic potential of the form $U_{\text{ext}} = \frac{1}{2}m\omega^2 r^2$, where m is the particle mass, r is the distance from the center of the trap, and $\varepsilon_0 = \hbar\omega$ is the oscillator energy. Assume for simplicity that the bosons have no internal degrees of freedom, either being spinless or selected by magnetic substate, and that mutual interaction can be neglected.

(a) Show that a continuous approximation to the density of states for a spinless particle confined by an isotropic three-dimensional harmonic potential takes the form

$$g(\varepsilon) = \frac{\varepsilon^2}{2\varepsilon_0^3}. \tag{3}$$

VMY Hint: Introducing dimensionless coordinate $\tilde{r} = r\sqrt{m\omega/\hbar}$ and momentum $\tilde{p} = p/\sqrt{m\hbar\omega}$, show that the energy of a particle in a harmonic trap can be written as $\varepsilon = \varepsilon_0(\tilde{r}^2 + \tilde{p}^2)/2 = \varepsilon_0 R^2/2$, where R is the radius in the 6-dimensional phase space of $\tilde{r}$ and $\tilde{p}$. Density of states can be obtained as $g = dN/d\varepsilon$. The number of states dN is proportional to the volume of a spherical shell of thickness dR in the 6-dimensional phase space. (The area of a 5-dimensional sphere is given in Appendix below.) The energy increment is $d\varepsilon = \varepsilon_0\, R\, dR$.

(b) Derive a relationship between the critical temperature of Bose-Einstein condensation T_c and the particle number N_c.

(c) Using the fact that the mean kinetic and potential energies are equal for a harmonic potential, deduce the mean value $\langle r^2 \rangle$ in terms of T, m, and ω.

Using this result, estimate the average density of bosons in the excited states of the trap. [Hint: Use the formula $V \sim \langle r^2 \rangle^{3/2}$ to estimate the volume V occupied by the Bose gas.] How does the critical density $n_c = N_c/V$ depend on temperature?

(d) *Numerical estimates - removed by VMY.*

(e) Under what conditions should your analysis be valid?

(f) *Added by VMY.* Discuss qualitatively whether Bose-Einstein condensation should happen in a harmonic potential in two dimensions and in one dimension. Compare with Problem 2.

Appendix. Possibly useful information:

The area of a 5-dimensional sphere is $S_5 = \pi^3 R^5$.

$$\int_0^\infty dx \frac{x^n}{e^x - 1} = n!\,\zeta(n{+}1), \qquad \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \zeta(2,\, 3,\, 4,\, 5) \approx \{1.64,\, 1.20,\, 1.08,\, 1.04\}$$