Homework #4 — PHYS 603 — Spring 2007
Deadline: **Tuesday, April 17, 2007, in class**

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Textbook: Gregory H. Wannier, *Statistical Physics*
Dover 1987 reprint of the 1966 edition, ISBN 0-486-65401-X

Do not forget to write your name and the homework number!

Each problem is worth 10 points.

Ch. 4 The Gibbs-Boltzmann Distribution

1. **Problem 4.2, Derive the Maxwell distribution $p(v)$ for the speed v.**

2. **Problem 4.3, Compare $\langle v\rangle$ with $\sqrt{\langle v^2\rangle}$**

3. **Problem 4.6, Calculate the most probable speed v_*.**

4. **Energy distribution $p(E)$ for the 3D Maxwell gas.**

   (a) Derive the probability density $p(E)$ for the distribution of energy E of an atom in a three-dimensional (3D) Maxwell gas. The definition of the probability density is $dP = p(E)\,dE$, where dP is the probability to have energy in the interval $(E, E + dE)$. **Hint:** Follow the discussion around Eq. (4.35) for one particle $n = 1$.

   (b) From the probability density $p(E)$ derived above, calculate the most probable energy E_* which maximizes $p(E)$. Compare E_* with $mv_*^2/2$, where v_* is the most probable speed maximizing $p(v)$, as derived in Problem 3. Are E_* and $mv_*^2/2$ equal? If not, can you explain why?

5. **Problem 4.7, Mean kinetic energy in a leak.**

   Hint: To calculate the mean kinetic energy for a leak in the x direction, calculate the energy flux $\langle Ev_x\rho(E)\rangle$ and divide it by the particle flux $\langle v_x\rho(E)\rangle$. Here $E = m(v_x^2+v_y^2+v_x^2)/2$ is the energy of an atom, $\rho[E(\boldsymbol{v})]$ is the Gibbs-Boltzmann distribution function, and the averaging $\langle\ldots\rangle$ is obtained by integrating over $dv_x\,dv_y\,dv_z$ under condition $v_x \geq 0$ representing particles moving to the right.

6. **Problem 4.9, Prove that $\langle v\rangle\langle 1/v\rangle \geq 1$ always.**

   Hint: Suppose you did N measurements of a variable v and found a set of values $\{v_i\}$. By the definition of the average, we have

$$\langle v\rangle = \frac{1}{N}\sum_{i=1}^{N} v_i, \qquad \left\langle \frac{1}{v}\right\rangle = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{v_j}. \tag{1}$$

Multiply $\langle v\rangle$ and $\langle 1/v\rangle$, and use the inequality

$$\frac{x}{y}+\frac{y}{x}\geq 2, \tag{2}$$

which is valid for any x and y.

Digression: The inequality $\langle v\rangle\langle 1/v\rangle \geq 1$ has some financial applications. Suppose you spend money M on a regular basis to buy stock S over a long time interval t_2-t_1, say as an investment for retirement. However, the stock price $v=dM/dS$ fluctuates in time t. Let us compare two strategies called the "dollar averaging" ($A) and the "share averaging" (SA).

In the dollar averaging strategy, one spends the same amount of money per unit time no matter what is the stock price. Then, the total amount of stock purchased is

$$S=\frac{M}{t_2-t_1}\int_{t_1}^{t_2}\frac{dt}{v(t)}=M\left\langle\frac{1}{v}\right\rangle_t, \tag{3}$$

where $\langle\ldots\rangle_t$ represents averaging over time. In the share averaging strategy, one buys the same amount of stock per unit time no matter what is the stock price. Then, the total money spent is

$$M=\frac{S}{t_2-t_1}\int_{t_1}^{t_2}dt\,v(t)=S\langle v\rangle_t. \tag{4}$$

Comparing the two strategies (3) and (4)

$$\left.\frac{S}{M}\right|_{\$A}=\left\langle\frac{1}{v}\right\rangle_t\geq\left.\frac{S}{M}\right|_{SA}=\frac{1}{\langle v\rangle_t}, \tag{5}$$

we see that dollar averaging is better, because it buys more stock for the same money. This is way dollar averaging is always recommended by retirement advisors.

7. **Problem 4.10, At what T a half of oscillators are excited?**

8. **Problem 4.13, A more general equipartition theorem.**

9. **Problem 4.14, The law of atmospheres for gravity.**

   Assume constant temperature everywhere, which may be not quite realistic. To calculate the numerical value, you would need to know the mass m of a molecule in the atmosphere, which is different for different molecules. So, do not calculate the numerical value, unless you wish to, and only derive a formula.

10. **Problem 4.16, The flux of particles in the 3D Maxwell gas.**

    See the Hint for Problem 5.

11. **UMD qualifier problem, January 2004: Probability distribution of money.**

    This problem explores an analogy between the Boltzmann-Gibbs probability distribution of energy in statistical physics and the probability distribution of money in a closed system of economic agents. Consider a system consisting of $N\gg 1$ economic agents.

At a given moment of time, each agent i has a non-negative amount of money $m_i \geq 0$ (debt is not permitted). As the agents engage in economic activity, money is constantly transferred between the agents in the form of payments. However, the total amount of money is conserved in binary transactions between agents: $m_i + m_j = m_i' + m_j'$. This condition is analogous to conservation of energy in collisions between atoms in a gas. We also assume that the system is closed, so the total amount of money in the system M is also conserved.

We wish to obtain a formula for the money distribution function $P(m)$ for the system in statistical equilibrium. It is defined so that $P(m)\,dm$ is the fraction of agents with money in the interval $[m, m+dm]$ and satisfies the standard normalization conditions:

$$\int_0^\infty P(m)\,dm = 1, \qquad \int_0^\infty m\,P(m)\,dm = \langle m \rangle = M/N, \tag{6}$$

where $\langle m \rangle = M/N$ is the average amount of money per agent.

- **(a)** Let us divide the money semi-axis $m \geq 0$ into equal intervals Δm and count the number of agents belonging to each interval: N_1, N_2, $N_3, \ldots$ Obviously $\sum_{r=1}^\infty N_r = N$. This configuration can be realized in many different ways by moving agents between the intervals while preserving the occupation numbers N_1, N_2, $N_3, \ldots$ Write a combinatorial formula for the number of ways W a given distribution N_1, N_2, $N_3, \ldots$ can be realized.
- **(b)** Using the Stirling approximate formula $\ln n! \approx n \ln n - n$ for $n \gg 1$, obtain an expression for $S = \ln W$, the entropy of the distribution.
- **(c)** Using the method of Lagrange multipliers, find the configuration N_r^* that maximizes entropy S under constraints that the total number of agents is conserved and the total amount of money is conserved:

$$\sum_{r=1}^\infty N_r = N, \qquad \sum_{r=1}^\infty m_r\,N_r = M. \tag{7}$$

- **(d)** Obtain the money distribution function $P(m)$ for the fraction of agents belonging to a given interval Δm: $P(m_r) = N_r^*/N$. Determine the values of the Lagrange multipliers from the normalization conditions (6).
- **(e)** Compare the obtained result for $P(m)$ with the Boltzmann-Gibbs formula for the probability distribution of energy $P(E)$ in physics. What is the analog of temperature in the system of economic agents? Compare with $\langle m \rangle$.