FINAL EXAM — PHYS 603 — Spring 2007 Professor Victor Yakovenko
Monday, May 14, 2007, 8–10 am Office: 2115 Physics

You may use the textbook, your notes, homework, and any other materials.

Do not forget to write your name!

Each problem is worth 10 points.

Please write the Boltzmann constant k_B explicitly in this exam.

1. **UMD qualifier problem, January 2000: Fluctuations in an *LC* circuit.**

   An interesting thermometer may be made by observing the voltage fluctuations in a circuit consisting of an ideal inductor L connected in parallel to an ideal capacitor C. The circuit is in thermal contact with a heat reservoir at temperature T.

   (a) Write down an expression for the electric energy stored in the capacitor C and magnetic energy stored in the inductor L. Show that this system is a harmonic oscillator with appropriate analogs of mass and spring constant. What is the natural frequency ω_0 of this oscillator?

   (b) Using classical statistical mechanics, deduce the root-mean-square voltage in the circuit $V_{\rm rms} = \sqrt{\langle V^2 \rangle}$.

   (c) Assuming that the oscillator energy states follow the quantum mechanical rule $E_n = (n + 1/2)\hbar\omega_0$, where ω_0 is the classical frequency of the oscillator, $\hbar$ is the Planck constant, and $n = 0, 1, 2, \ldots$, deduce the temperature dependence of $V_{\rm rms}$.

   (d) Sketch $V_{\rm rms}$ vs. T. Evaluate the low and high T behavior of $V_{\rm rms}$ and compare with the classical result. For what range of T would this thermometer be useful?

2. **UMD qualifier problem, January 2007: Fluctuations of a wire.**

   A wire of length l and mass per unit length μ is fixed at both ends and tightened to a tension τ. In this problem, you will use the principles of classical statistical mechanics to find the transverse RMS fluctuations of the midpoint of the wire when it is in equilibrium with a heat bath at temperature T.

   (a) Let us describe the transverse displacement of wire by the function $y(x,t)$ of the coordinate x along the wire and time t.

   Show that the energy of the wire for small displacements is given by the following expression

   $$E = \frac{1}{2}\int_0^l dx \left[\mu\left(\frac{\partial y}{\partial t}\right)^2 + \tau\left(\frac{\partial y}{\partial x}\right)^2\right], \qquad (1)$$

   where the first term represents kinetic energy of transverse motion, and the second term represent elastic energy due to stretching of the wire.

   [Hint: The elastic energy can be written as $\tau(\tilde{l} - l)$, where $\tilde{l} = \int\sqrt{dx^2 + dy^2} = \int dx\sqrt{1 + (\partial y/\partial x)^2}$ is the length of the stretched wire. Expanding in $\partial y/\partial x$ gives the second term in Eq. (1).]

   Write down the equation of motion for $y(x,t)$.

(b) Show that the normal modes of vibrations of the wire have the form

$$y_n(x,t) = A_n \sin(k_n x - \omega_n t). \tag{2}$$

Find the permitted values of k_n for symmetric modes. (The antisymmetric modes will not move the midpoint of the wire at all.)

Find a relationship between the frequency ω_n and wavenumber k_n.

(c) Use Eqs. (1) and (2) to show that the energy of the n-th normal mode is

$$E_n = \frac{\tau \pi^2 n^2}{2l} A_n^2, \tag{3}$$

where A_n is the maximal displacement of the n-th mode.

(d) By taking into account the number of degrees of freedom in each normal mode, relate E_n to the temperature T.

(e) Using these results, find an expression for the time-averaged mean-square displacement at the center of the wire $y_{rms}^2 = \langle y^2(l/2,t)\rangle_t$.

(f) Using the handy formula

$$\sum_m^\infty \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}, \tag{4}$$

show that

$$y_{rms} = \frac{1}{2}\sqrt{\frac{l k_B T}{\tau}} \tag{5}$$

3. UMD qualifier problem, August 2006: Spin 1/2 in magnetic field.

A system of N fixed particles with spin 1/2 and magnetic moment mu_0 is in a static, uniform magnetic field B. The spins interact with the applied field but otherwise are essentially free.

(a) Express the energy of the system as a function of its total magnetic moment and applied field.

(b) Find the total magnetic moment and the energy assuming that the system is in thermal equilibrium at temperature T.

(c) Find the heat capacity and the entropy of the system under these same conditions.