Homework #9 (Make-up for Midterm Exam) Professor Victor Yakovenko
Deadline: **Tuesday, November 21, 2006, in class** PHYS 374 — Fall 2006

You may attach your graded exam and a cover sheet explaining which problem scores should be counted from the graded exam and which from the new submission.

Do not forget to write your name!

1. Electron velocity in the hydrogen atom [6 points (2 points per part)]

(a) Using dimensional analysis, derive a formula (in the Gaussian system of units) for the electron velocity v_B in the hydrogen atom. You may argue that it can depend on the electron charge e, the electron mass m, and the Planck constant $\hbar$.

(b) Compare the electron velocity v_B in the hydrogen atom with the speed of light c. Express the dimensionless ratio v_B/c in terms of a dimensionless combination of fundamental constants. What is the name and the numerical value of this ratio?

(c) On the basis Part 1b, do you think the non-relativistic approximation is appropriate for description of the hydrogen atom? Does this justify not including the speed of light c among the relevant parameters in Part 1a?

Numerical information. In the Gaussian system of units: $\alpha = e^2/\hbar c = 1/137$.
In the SI system of units: $\alpha = e^2/\hbar c 4\pi\epsilon_0 = 1/137$.

2. Taylor series expansion for potential energy [8 points]

A charge q is located near the origin $(0,0,0)$ and is surrounded by four charges Q located at the points $(\pm a, 0, 0)$ and $(0, \pm a, 0)$. Make a sketch of the problem.

Write down the Coulomb potential energy $V(x, y, z)$ as a function of the coordinates (x, y, z) of the charge q for the fixed positions of the charges Q.

Using the appropriate Taylor series, expand $V(x, y, z)$ to the lowest non-vanishing order in powers of x, y, and z (linear or quadratic) for small $|x|, |y|, |z| \ll a$. By the symmetry of the problem, do you expect any odd powers of x, y, or z in the expansion?

On the basis of your result, is the point $(0,0,0)$ an equilibrium point for charge q? Is it stable or unstable? Explain why.

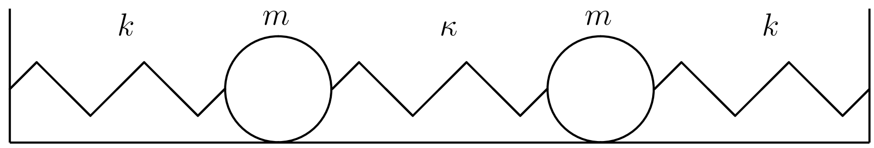
Using your expansion result for $V(x, y, z)$, calculate the Laplacian $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V(x, y, z)$ and show that it is equal to zero.

The potential energy of interaction between two charges located at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ is

$$V = k\frac{q_1\, q_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} = k\frac{q_1\, q_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}. \tag{1}$$

3. Two coupled oscillators

Consider two coupled oscillators, as shown in the figure. Two masses m are attached to the walls by the springs of rigidity k and are connected with each other by another spring of rigidity κ. The masses can slide horizontally without friction.



(a) [**2 points**] Write the equations of motion for small horizontal displacements x_1 and x_2 of the two masses from the equilibrium positions in the 2×2 matrix form.

(b) [**4 points**] Taking the time dependence in the form

$$\boldsymbol{v}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{-i\omega t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{2}$$

determine the eigenfrequencies ω_+ and ω_-, and the corresponding eigenvectors $\boldsymbol{v}_+$ and $\boldsymbol{v}_-$ of the normal modes.

(c) [**2 points**] Describe the symmetry of the obtained normal modes. Are the eigenvectors orthogonal? Which mode has the lower frequency? Does the lower frequency depend on κ and why?

(d) [**4 points**] Consider the initial conditions at time $t = 0$, where the first mass has a small displacement x_0, whereas the second mass is not displaced:

$$\boldsymbol{v}(t = 0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \tag{3}$$

Represent the initial vector (3) as a superposition of the eigenvectors $\boldsymbol{v}(t = 0) = c_+ \boldsymbol{v}_+ + c_- \boldsymbol{v}_-$ with some coefficients c_+ and c_-, and find the time evolution of the system for these initial conditions as

$$\boldsymbol{v}(t) = c_+ \boldsymbol{v}_+ e^{-i\omega_+ t} + c_- \boldsymbol{v}_- e^{-i\omega_- t}. \tag{4}$$

Do not forget to take the real part at the end.

(e) [**2 points**] Consider the case, where the two oscillators are weakly coupled, i.e. $\kappa \ll k$. Using your solution of Part 3d, qualitatively describe the time evolution of the system. Make a sketch of $x_1(t)$ and $x_2(t)$. What happens to the energy of oscillations, initially localized in the first oscillator, as the time goes on?

In quantum mechanics, this evolution is called the Rabi oscillations.

4. Dirac delta-function [**8 points**]

Calculate the integral

$$D = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(x^2 - y^2)\, \delta(x^2 + y^2 - R^2)\, f(x, y)\, dx\, dy, \tag{5}$$

where δ is the Dirac delta-function, and $f(x, y)$ is some generic function.

Make a sketch in the (x, y) plane to show geometrically where the conditions of the first and the second delta-functions are satisfied.