

Textbook: Roel Snieder, *A Guided Tour of Mathematical Methods for the Physical Sciences*, Cambridge University Press, 2nd edition, 2004, ISBN 0-521-83492-9

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Do not forget to write your name and the homework number!

Ch. 15 Fourier Analysis

1. Parseval's theorem

Let us consider a (complex) function $f(t)$, such that $|f(t)| \rightarrow 0$ at $t \rightarrow \pm\infty$, and its Fourier transform $F(\omega)$

$$F(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} f(t), \quad f(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} F(\omega). \quad (1)$$

Notice that my definition (1) differs from the textbook Eqs. (15.42) and (15.43) by the factor 2π in a different place. Use my definition (1) in this homework.

- (a) [4 points] Prove Parseval's theorem (see discussion of Eq. (15.53) in the textbook):

$$\int_{-\infty}^{+\infty} dt |f(t)|^2 = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} |F(\omega)|^2. \quad (2)$$

This identity shows that the norm of a function $\|f\| \equiv \sqrt{\langle f|f \rangle}$ can be equivalently calculated in either time representation or frequency representation.

- (b) [4 points] Prove the following identity:

$$\int_{-\infty}^{+\infty} dt \left| \frac{df(t)}{dt} \right|^2 = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \omega^2 |F(\omega)|^2. \quad (3)$$

2. [4 points] The Cauchy–Bunyakovski–Schwarz inequality

Consider two arbitrary (complex) vectors \mathbf{A} and \mathbf{B} . Prove the Cauchy–Bunyakovski–Schwarz inequality:

$$\langle \mathbf{A}|\mathbf{A} \rangle \langle \mathbf{B}|\mathbf{B} \rangle \geq |\langle \mathbf{A}|\mathbf{B} \rangle|^2 \quad \text{or} \quad \|\mathbf{A}\| \|\mathbf{B}\| \geq |\langle \mathbf{A}|\mathbf{B} \rangle| \quad (4)$$

Suggestions for the proof. Consider the component of \mathbf{A} perpendicular to \mathbf{B} : $(\mathbf{A} - \gamma\mathbf{B})$, where $\gamma = \langle \mathbf{B}|\mathbf{A} \rangle / \langle \mathbf{B}|\mathbf{B} \rangle$, see Eq. (13.3) in the textbook. Obviously, the square of this component is non-negative:

$$\langle \mathbf{A} - \gamma\mathbf{B}|\mathbf{A} - \gamma\mathbf{B} \rangle \geq 0 \quad (5)$$

Expanding the product in the inequality (5) and substituting the expression for γ , obtain the inequality (4). Do not forget complex conjugation in the definition of the scalar product: $\langle \mathbf{B}|\mathbf{A} \rangle = \mathbf{B}^* \cdot \mathbf{A} = \langle \mathbf{A}|\mathbf{B} \rangle^*$, see Eq. (15.20) in the textbook.

This proof shows that the equality in (4) is obtained when the two vectors are parallel, i.e. proportional $\mathbf{A} = c\mathbf{B}$.

In application to functions (see Eq. (15.20) in the textbook), the inequality (4) becomes

$$\int |f(x)|^2 dx \int |g(x)|^2 dx \geq \left| \int f^*(x) g(x) dx \right|^2. \quad (6)$$

3. [4 points] The Heisenberg uncertainty relation

The Heisenberg uncertainty relation is usually associated with quantum mechanics (it contains \hbar). On the other hand, it can be rewritten without \hbar as the uncertainty relation between wave vector and coordinate, or frequency and time:

$$\Delta k \Delta x \geq 1/2, \quad \Delta \omega \Delta t \geq 1/2. \quad (7)$$

The inequalities (7) are well known in such classical disciplines as optics, electrical engineering, data transmission, etc. They are general inequalities relating the width of a function and the width of its Fourier transform.

Let us show that computer circuits and music players obey the (second) uncertainty relation (7). Let us denote the voltage as a function of time t somewhere in the electric circuit by $f(t)$. For concreteness, let us assume that $f(t)$ has the shape of a pulse vanishing at $t \rightarrow \pm\infty$. Without loss of generality, we can normalize $f(t)$ to 1:

$$\int |f(t)|^2 dt = 1. \quad (8)$$

Arrival of the pulse signifies arrival of a bit of information. However, since $f(t)$ has some width in time, there is uncertainty in the arrival time. It is natural to define the center t_0 and the width Δt (the uncertainty of the arrival time) of a pulse as

$$t_0 = \int t |f(t)|^2 dt, \quad \Delta t = \sqrt{\int (t - t_0)^2 |f(t)|^2 dt}. \quad (9)$$

Similarly, we define the center ω_0 and the width $\Delta\omega$ (the frequency uncertainty) of a signal in the frequency domain as

$$\omega_0 = \int \omega |F(\omega)|^2 \frac{d\omega}{2\pi}, \quad \Delta\omega = \sqrt{\int (\omega - \omega_0)^2 |F(\omega)|^2 \frac{d\omega}{2\pi}}, \quad (10)$$

where $F(\omega)$ is given by Eq. (1).

By shifting the time and frequency variables of integration as $(t-t_0) \rightarrow t$ and $(\omega-\omega_0) \rightarrow \omega$, we can eliminate t_0 and ω_0 from the integrals. Thus, to prove (7), it is sufficient to prove the following inequality:

$$(\Delta\omega)^2 (\Delta t)^2 = \int t^2 |f(t)|^2 dt \int \omega^2 |F(\omega)|^2 \frac{d\omega}{2\pi} \geq \frac{1}{4}. \quad (11)$$

Prove the inequality (11).

Suggestions for the proof. Using (3) and (6), rewrite the left-hand side of (11) as

$$\int t^2 |f(t)|^2 dt \int \left| \frac{df(t)}{dt} \right|^2 dt \geq \left| \int t f^*(t) \frac{df(t)}{dt} dt \right|^2. \quad (12)$$

If $f(t)$ is real, then the integral in the right-hand side of (12) can be written as $\int t (df/dt)^2 dt/2$ and then integrated by parts. If $f(t) = f'(t) + if''(t)$ is complex, where f' and f'' are the real and imaginary parts, we can write the right-hand-side integral in (12) as

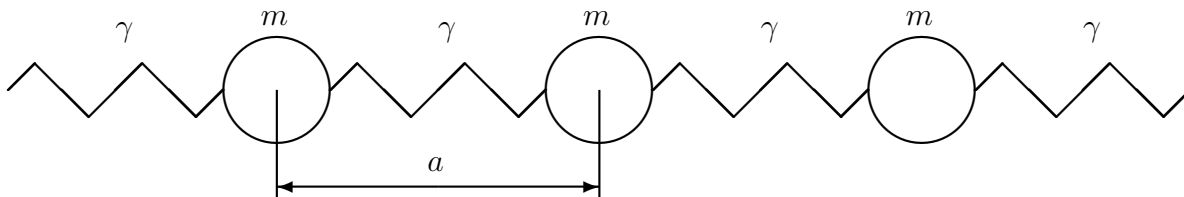
$$\left| \int t f^*(t) \frac{df(t)}{dt} dt \right| \geq \int t \left(f'(t) \frac{df'(t)}{dt} + f''(t) \frac{df''(t)}{dt} \right) dt \quad (13)$$

and then apply the integration-by-parts procedure to f' and f'' . Using the normalization condition (8) finally proves the inequality (11).

The inequalities (7) state that, in order to reproduce fast time variations of a signal with high fidelity, the system must have sufficiently wide bandwidth in frequency. That is why optical fiber, utilizing the frequency range of visible light, can transmit digital data much faster than copper wires, whose range is limited to much lower frequencies.

4. A chain of oscillators

Consider an infinite chain of atoms with the masses m connected by springs of the rigidity γ with the spacing a between the equilibrium positions of the atoms:



- (a) [2 points] Let us use $u(n, t)$ denote a small displacement of the atom with the number n from its equilibrium position. Write Newton's equations of motion for $u(n, t)$ in terms of the displacements $u(n-1, t)$ and $u(n+1, t)$ of the neighboring atoms.
- (b) [4 points] Determine the normal modes and the eigenfrequencies of the system. To accomplish this, diagonalize Newton's equations of motion obtained in Part 4a by doing a Fourier transform, which amounts to substituting the wave solutions

$$u(n, t) = e^{-i\omega t} e^{ikna} \tilde{u}(k) \quad (14)$$

into the equations of motion. In Eq. (14), ω and k are the frequency and the wave vector of a normal mode of oscillations. Given that $x_n = na$ is the equilibrium coordinate of the atom n , Eq. (14) corresponds to the Fourier transform from $u(x_n)$ to $\tilde{u}(k)$.

By substituting (14) into the equations of motion, obtain a relation $\omega(k)$ between the wave vector and the frequency, which is called the dispersion relation. Sketch a plot of $\omega(k)$. What is $\omega(k=0)$? Give physical reasons why.

- (c) [4 points] Consider small wavevectors $ka \ll 1$ with the wavelength $l = 2\pi/k \gg a$ much longer than a . Expand $\omega(k)$ to the lowest non-vanishing power of k . Show that, for $ka \ll 1$, the dispersion relation is linear: $\omega = sk$, where s is the wave speed. Express s in terms of k , m , and a . Check that the dimensionality is correct.

5. Elastic waves in three-dimensional crystals

Let us generalize Problem 4 to three dimensions. Consider a three-dimensional cubic lattice of atoms with the masses m connected by springs of the rigidity γ with the spacing a between the equilibrium positions of the atoms along the x , y , and z directions.

- (a) [4 points] The equilibrium positions $\mathbf{r}_n = a\mathbf{n}$ of the atoms are now labeled by the three-dimensional vector $\mathbf{n} = (n_x, n_y, n_z)$ with the integer components. Let us use $\mathbf{u}(\mathbf{n}, t)$ to denote the displacement vector of the atom \mathbf{n} from its equilibrium position. Write Newton's equations of motion for $u_x(\mathbf{n}, t)$, $u_y(\mathbf{n}, t)$, and $u_z(\mathbf{n}, t)$.
- (b) [4 points] Determine the normal modes and the eigenfrequencies of the system. To accomplish this, diagonalize Newton's equations of motion obtained in Part 5a by doing a Fourier transform, which amounts to substituting the plane wave solutions

$$\mathbf{u}(\mathbf{n}, t) = e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{n}a} \tilde{\mathbf{u}}(\mathbf{k}) \quad (15)$$

into the equations of motion. Eq. (15) corresponds to the Fourier transform from $\mathbf{u}(\mathbf{r}_n)$ to $\tilde{\mathbf{u}}(\mathbf{k})$.

By substituting (15) into the equations of motion, obtain a dispersion relation $\omega(\mathbf{k})$ between the wave vector and the frequency. What is $\omega(\mathbf{k} = 0)$?

- (c) [4 points] The elastic waves discussed above describe propagation of sound in crystals and are also called phonons. Show that the speed s obtained in Part 4c can be written as the speed of sound

$$s = \sqrt{\frac{\partial P}{\partial \rho}}, \quad (16)$$

where P is pressure, and ρ is mass density.

Suggestions for the proof. Consider a volume of the length L in the x direction and the area A in the perpendicular direction. Suppose we compress the crystal in the x direction, so that the length becomes $L - \delta L$. Each spring becomes compressed in the x direction by the distance $a\delta L/L$, which produces the total elastic force $\delta F = N\gamma a\delta L/L$, where $N = A/a^2$ is the total number of chains in the area A . On the other hand, density becomes $\rho' = (m/a^3)L/(L - \delta L)$. Expand the density change $\delta\rho = \rho' - \rho$ to the first order in δL and calculate $\delta P/\delta\rho = \delta F/A\delta\rho$.