Homework #7 — PHYS 374 — Fall 2006 Professor Victor Yakovenko
Deadline: **Tuesday, November 7, 2006, in class** Office: 2115 Physics

Textbook: Roel Snieder, *A Guided Tour of Mathematical Methods for the Physical Sciences*, Cambridge University Press, 2nd edition, 2004, ISBN 0-521-83492-9

Web page: http://www2.physics.umd.edu/~yakovenk/teaching/

Do not forget to write your name and the homework number!

Ch. 15 Fourier Analysis

1. **Fourier transform of the sawtooth wave [6 points]**

   Calculate the Fourier coefficients a_n and b_n, given by Eqs. (15.7) and (15.8) in the textbook, for the following function:

   $$f(x) = x, \quad -L \le x \le L \tag{1}$$

   and continued periodically $f(x + 2L) = f(x)$.

   Sketch the function $f(x)$ over several periods. Use the symmetry to explain why one of the sets of the Fourier coefficients a_n or b_n vanishes identically.

   Optional part: Using Mathematica or a similar program, make plots of the sum (15.2) with the coefficients a_n and b_n that you calculated. Keep a finite number of terms in the sum and observe how the sum converges to the sawtooth wave as you increase the number of terms in the sum.

2. **Fourier transform of the rectified wave [6 points]**

   Calculate the Fourier coefficients a_n and b_n, given by Eqs. (15.7) and (15.8) in the textbook, for the following function:

   $$f(x) = \cos(\pi x/2L), \quad -L \le x \le L \tag{2}$$

   and continued periodically $f(x + 2L) = f(x)$.

   Sketch the function $f(x)$ over several periods. Use the symmetry to explain why one of the sets of the Fourier coefficients a_n or b_n vanishes identically.

   Optional part: Using Mathematica or a similar program, make plots of the sum (15.2) with the coefficients a_n and b_n that you calculated. Keep a finite number of terms in the sum and observe how the sum converges to the rectified wave as you increase the number of terms in the sum.

3. **Fourier transform of exponential functions**

   (a) [**6 points**] Calculate Fourier transforms of the following functions:

$$f(x) = \Theta(x)\, e^{-ax}, \tag{3}$$
$$f(x) = \Theta(-x)\, e^{ax}, \tag{4}$$
$$f(x) = e^{-a|x|}, \tag{5}$$

where a is a positive real number, and $\Theta(x)$ is the step function:

$$\Theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \tag{6}$$

(b) [**3 points**] Using your Fourier transform of the function (5), prove the following relation:

$$\lim_{a\to 0} \int dx\, e^{-ixk}\, e^{-a|x|} = 2\pi\, \delta(k), \tag{7}$$

i.e. the Fourier transform of a constant function (when $a = 0$) is the delta-function.

4. Response function of an *RL* circuit

Consider a circuit consisting of an inductance L and a resistance R connected in series. The input voltage $V_{in}(t)$ is applied to the series of L and R, whereas the output voltage $V_{out}(t)$ is taken from the resistor R. The circuit equations are

$$V_{in}(t) = L\frac{dI(t)}{dt} + R\, I(t) \tag{8}$$
$$V_{out}(t) = R\, I(t) \tag{9}$$

(a) [**3 points**] Equations (8) and (9) are written in the time domain. Fourier-transform these equation to the frequency domain, i.e. obtain equations relating the Fourier transforms

$$\tilde{V}_{in}(\omega) = \int dt\, e^{i\omega t}\, V_{in}(t), \quad \tilde{V}_{out}(\omega) = \int dt\, e^{i\omega t}\, V_{out}(t), \quad \tilde{I}(\omega) = \int dt\, e^{i\omega t}\, I(t). \tag{10}$$

In particular, show that

$$\frac{dI(t)}{dt} \to -i\omega\tilde{I}(\omega). \tag{11}$$

(b) [**3 points**] Eliminate $\tilde{I}(\omega)$ from the equations obtained in Part 4a and obtain a relation between the input and output voltages in the frequency domain:

$$\tilde{V}_{out}(\omega) = \tilde{G}(\omega)\, \tilde{V}_{in}(\omega), \tag{12}$$

where $\tilde{G}(\omega)$ is a response function. Derive an explicit expression for $\tilde{G}(\omega)$.

(c) [**3 points**] Fourier-transform Eq. (12) to the time domain and obtain a relation between the input and output voltages in the form

$$V_{out}(t) = \int_{-\infty}^{t} dt'\, G(t-t')\, V_{in}(t'). \tag{13}$$

Derive an explicit expression for $G(t)$ using the results of Problem 3a. Make sure that dimensionality is correct in your final result.

5. Transmission line and filtering

(a) [**2 points**] Suppose an input signal $V_{in}(t)$ is sent through an ideal transmission line, so the output signal $V_{out}(t)$ emerges with a delay time t_0:

$$V_{out}(t) = V_{in}(t - t_0). \tag{14}$$

Show that, in this case, the response function has the following forms in the time and frequency domains:

$$G(t) = \delta(t - t_0), \quad \tilde{G}(\omega) = \int dt\, e^{i\omega t}\, G(t) = e^{i\omega t_0}. \tag{15}$$

(b) [**4 points**] Suppose the input signal is a Gaussian function

$$V_{in}(t) = V_0 \exp(-t^2/2\tau^2). \tag{16}$$

This is a pulse of the width τ, which may represent, for example, a digital bit. Calculate the Fourier transform $\tilde{V}_{in}(\omega)$ of Eq. (16) to the frequency domain.

(c) [**2 points**] Suppose the transmission line with a limited bandwidth in frequency (like a phone line). Let us model the limited bandwidth by a Gaussian function of ω with the width ω_0 and take the response function to be

$$\tilde{G}(\omega) = e^{i\omega t_0} \exp(-\omega^2/2\omega_0^2). \tag{17}$$

Using the result of Part 5b and Eq. (17), calculate the output signal $\tilde{V}_{out}(\omega)$ in the frequency domain:

$$\tilde{V}_{out}(\omega) = \tilde{G}(\omega)\, \tilde{V}_{in}(\omega). \tag{18}$$

(d) [**4 points**] Fourier transform Eq. (18) to the time domain and obtain $V_{out}(t)$:

$$V_{out}(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \tilde{V}_{out}(\omega). \tag{19}$$

What is the shape of the function $V_{out}(t)$? How does the width of $V_{out}(t)$ compare with the width of $V_{in}(t)$? Sketch both $V_{out}(t)$ and $V_{in}(t)$ on the same plot (do not forget the time delay t_0).

(e) [**2 points**] Suppose we send a series of input pulses (16) with the time interval Δt between the pulses. The input pulses can be well resolved in $\Delta t \geq \tau$. Suppose we send the pulses through the transmission line with the limited bandwidth $\omega_0 \ll 1/\tau$. What is the lower limit on the time interval Δt between the pulses such that we can well resolve the pulses in the output? Observe that the bandwidth ω_0 of the transmission line limits how fast we can send digital pulses through.