Homework #6 — PHYS 374 — Fall 2006
Deadline: **Tuesday, October 24, 2006, in class**

Professor Victor Yakovenko
Office: 2115 Physics

Textbook: Roel Snieder, *A Guided Tour of Mathematical Methods for the Physical Sciences*, Cambridge University Press, 2nd edition, 2004, ISBN 0-521-83492-9

Web page: http://www2.physics.umd.edu/~yakovenk/teaching/

Do not forget to write your name and the homework number!

Ch. 13 Linear Algebra

1. **Two-dimensional oscillator subject to the Coriolis force**

   Consider a two-dimensional oscillator subject to the Coriolis force, which was discussed in class. Newton's equations of motion are

   $$\ddot{\boldsymbol{r}} = -\omega_0^2\,\boldsymbol{r} - 2\boldsymbol{\Omega}\times\dot{\boldsymbol{r}}, \tag{1}$$

   where $\boldsymbol{r} = (x, y)$ is the two-dimensional coordinate, ω_0 is the natural frequency of the oscillator, and $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{z}}$ is the angular velocity of the rotating reference frame.

   (a) [**8 points**] Write equations of motion for x and y in the 2×2 matrix form. Find exact eigenfrequencies ω_+ and ω_- and the corresponding eigenvectors of the normal modes. (Do not assume that $\Omega \ll \omega_0$, as we did in class.) Consider only positive eigenfrequencies $\omega_+ > 0$ and $\omega_- > 0$, because negative solutions do not give any additional information.

   Sketch a plot of the eigenfrequencies ω_+ and ω_- as functions of Ω from $\Omega = 0$ to $\Omega \gg \omega_0$.

   Describe physical meaning of the eigenvectors and their evolution in time.

   (b) [**4 points**] Consider periodic circular motion with a frequency ω. Using Newton's equation of motion (1) for circular motion, find an equation for ω.

   Consider the case $\Omega \gg \omega_0$ and find two solutions ω_+ and ω_- of this equation using the appropriate approximations.

   What is the physical meaning of the solution with the greater frequency ω_+? Have you encountered this kind of motion for a particle in a magnetic field? How is it called?

   What is the physical meaning of the solution with the lower frequency ω_-? What happens to this motion in the limit $\Omega \to \infty$?

Complex Numbers

2. **Hyperbolic functions and the Lorentz transformation**

(a) [**3 points**] Using the formula $e^{i\varphi} = \cos\varphi + i\sin\varphi$, show that

$$\cos(i\theta) = \cosh\theta, \quad \sin(i\theta) = i\sinh\theta, \quad \tan(i\theta) = i\tanh\theta. \tag{2}$$

The definitions of the hyperbolic functions are given in Problem 2 of Homework 2. Eq. (2) shows that hyperbolic functions are trigonometric functions taken at the imaginary argument.

(b) [**4 points**] Prove the following identities for the hyperbolic functions

$$\cosh^2\theta - \sinh^2\theta = 1, \qquad \tanh(\theta_1 + \theta_2) = \frac{\tanh\theta_1 + \tanh\theta_2}{1 + \tanh\theta_1 \tanh\theta_2}. \tag{3}$$

Read my handout about the Lorentz transformation, especially the last page:
`http://www2.physics.umd.edu/~yakovenk/teaching/Lorentz.pdf`.
It shows that the Lorentz transformation is described by a hyperbolic matrix, representing rotation over imaginary angle.

Ch. 14 Dirac Delta Function

3. Electron tunneling in a bilayer [8 points]

Let us consider a bilayer, i.e. two closely spaced semiconducting layers, where electrons can tunnel (make transitions) from one layer to another with a small amplitude.

Electron motion is characterized by the two-dimensional momentum (p_x, p_y). A low temperature, all interesting phenomena happen at the Fermi line. It is defined by the equation $(p_x^2 + p_y^2)/2m = E_F = p_F^2/2m$, where E_F is the Fermi energy, and p_F is the Fermi momentum.

When a magnetic field is applied parallel to the layers, it shifts electron momentum in one layer relative to another by a vector q_x, which we take to be in the x direction.

When electrons tunnel between the layers, they must conserve energy and momentum and belong to the Fermi lines of both layers. Thus, the probability of tunneling between the layers is proportional to the factor

$$D = \iint \delta\left(\frac{p_x^2 + p_y^2 - p_F^2}{2m}\right) \delta\left(\frac{(p_x - q_x)^2 + p_y^2 - p_F^2}{2m}\right) dp_x\, dp_y. \tag{4}$$

Calculate D. Sketch a plot of D as a function the dimensionless parameter q_x/p_F. Interpret the results geometrically.

The factor D determines interlayer conductivity, which was be measured experimentally and compared with theory in the following papers. The pdf files of these papers are accessible from any computer with the campus IP address; otherwise they require subscription.

- J. P. Eisenstein, T. J. Gramila, L. N. Pfeiffer, and K. W. West, "Probing a two-dimensional Fermi surface by tunneling", Physical Review B **44**, 6511 (1991), `http://link.aps.org/abstract/PRB/v44/p6511`.
- J. A. Simmons, S. K. Lyo, J. F. Klem, M. E. Sherwin, and J. R. Wendt, "Submicrometer control of two-dimensional–two-dimensional magnetotunneling in double-well heterostructures", Physical Review B **47**, 15741 (1993), `http://link.aps.org/abstract/PRB/v47/p15741`.

The second paper studies the case where the Fermi momenta $p_F^{(1,2)}$ of the two layers are different.