Homework #4 — PHYS 374 — Fall 2006 Professor Victor Yakovenko
Deadline: **Tuesday, October 10, 2006, in class** Office: 2115 Physics

Textbook: Roel Snieder, *A Guided Tour of Mathematical Methods for the Physical Sciences*, Cambridge University Press, 2nd edition, 2004, ISBN 0-521-83492-9

Web page: http://www2.physics.umd.edu/˜yakovenk/teaching/

Do not forget to write your name and the homework number!

Ch. 4.2 Changing Coordinate Systems

1. **Combination of two rotation transformations**

   (a) [**2 points**] Consider a two-dimensional reference frame $(\hat{x}, \hat{y})$ and another reference frame $(\hat{x}', \hat{y}')$ rotated relative to the first one by the angle φ_1.

   Write down a 2×2 matrix $\mathbf{M}_{\varphi_1}$ that expresses the coordinates of a vector (u'_x, u'_y) with respect to the second frame in terms of the coordinates (u_x, u_y) with respect to the first frame:

   $$\boldsymbol{u}' = \mathbf{M}_{\varphi_1}\boldsymbol{u}. \tag{1}$$

   (b) [**2 points**] Consider a reference frame $(\hat{x}'', \hat{y}'')$ rotated relative to the second reference frame $(\hat{x}', \hat{y}')$ by the angle φ_2.

   Write down a 2×2 matrix $\mathbf{M}_{\varphi_2}$ that expresses the coordinates of a vector (u''_x, u''_y) with respect to the third frame in terms of the coordinates (u'_x, u'_y) with respect to the second frame:

   $$\boldsymbol{u}'' = \mathbf{M}_{\varphi_2}\boldsymbol{u}'. \tag{2}$$

   Sketch the basis vectors of all three reference frames, showing the corresponding angles of rotation.

   (c) [**2 points**] Show that the matrix expressing the coordinates of a vector (u''_x, u''_y) with respect to the third frame in terms of the coordinates (u_x, u_y) with respect to the first frame can be written as a product of the two previously derived matrices:

   $$\boldsymbol{u}'' = \mathbf{M}_{\varphi_2}\mathbf{M}_{\varphi_1}\boldsymbol{u}. \tag{3}$$

   Multiply the matrices in Eq. (3) and compare the result with the matrix $\mathbf{M}_{\varphi_2+\varphi_1}$ describing rotation by the angle $\varphi_2 + \varphi_1$. Using trigonometric identities, show that

   $$\mathbf{M}_{\varphi_2+\varphi_1} = \mathbf{M}_{\varphi_2}\mathbf{M}_{\varphi_1}. \tag{4}$$

   Eq. (4) shows that rotation operations form a *group*: an algebraic structure where a product of two elements (rotations) produces another element (combined rotation) belonging to the group. The matrices M_φ represent rotation operations, so they form a *representation* of the group of rotations. Group theory play important role in physics, particularly in quantum mechanics.

Ch. 13.3 Coriolis Force and Centrifugal Force

2. Rotation of the Foucault pendulum

(a) [**3 points**] Does the oscillation plane of the Foucault pendulum rotates clockwise or counterclockwise when viewed down from the top of the pendulum? Give the answers for the northern hemisphere, southern hemisphere, and equator.

(b) [**3 points**] Give a formula for the angle of rotation α of the oscillation plane during one day. Plot the rotation angle α vs. the polar angle θ. (The angle θ is between the directions to the North Pole and to the pendulum position from the center of the Earth.) Where the rotation is the strongest?

(c) [**3 points**] The Foucault pendulum is installed in the lobby near the big lecture halls in the Physics Building, close to our classroom 1402.

Write down positions of the oscillation plane of the pendulum at a certain time and 24 hours later. Determine the angle of rotation α of the oscillation plane during one day. Using these data, calculate the angle θ for College Park (called co-latitude, see the beginning of Ch. 4.) Look up the approximate geographic latitude of College Park and compare with your calculation.

Keep in mind that the plane of rotation can be characterized by two complimentary angles, e.g. 30° and $30^\circ + 180^\circ = 210^\circ$, because the pendulum has two extremal positions. To get consistent angles, you can record a number on the side of the pendulum ball and use the same side in both measurements. If you have not done this, use common sense to select the appropriate angle. Is α expected to be close to 0°, 360°, or 180° for College Park?

3. Newton's equations in the rotating reference frame

(a) [**1 point**] A object of mass m, attached by a string of length r, rotates around a certain point with the angular velocity ω_1 (relative to the laboratory reference frame). Write down Newton's equations of motion $\boldsymbol{F} = m\boldsymbol{a}$ in the laboratory (inertial) reference frame and determine the tension force in the string.

(b) [**4 point**] Let us consider the same problem in a reference frame rotating with the angular velocity ω_2 (relative to the laboratory reference frame) around the same axis. (Unlike in our discussion in class, here the reference frame rotates with a *different* angular velocity than the object.)

How would you characterize the motion of the object relative to this rotating reference frame? Determine the velocity $\boldsymbol{v}_{\rm rot}$ and acceleration $\boldsymbol{a}_{\rm rot}$ of the object relative to this rotating reference frame.

Write down Newton's equations of motion in the form (13.30) in the rotating reference frame and verify that they are indeed satisfied. (The force $\boldsymbol{F}$ in these equations is the tension force of the string obtained in Part 3a.)

4. Defection of free-falling bodies [6 points]

(a) [**3 points**] Using Eq. (13.30), show that the acceleration $\boldsymbol{a}$ of a freely falling body relative to the (rotating) Earth is

$$\boldsymbol{a} = \boldsymbol{g} - 2\boldsymbol{\Omega} \times \boldsymbol{v}. \tag{5}$$

(Here the acceleration and velocity are relative to the rotating reference frame attached to the Earth. The subscript "rot" has been dropped to simplify notation.) The acceleration $\boldsymbol{g}$ includes both the gravitational force and the centrifugal force. A mass on a string would hang along this vector $\boldsymbol{g}$, and we take this direction as the vertical direction.

(b) [**3 points**] Consider an object dropped from a height h. Because of the Coriolis term in Eq. (5), does the object deflect from the vertical direction to the North, South, East, or West? Give the answer for the northern hemisphere, southern hemisphere, and equator. Sketch the magnitude of displacement vs. the polar angle θ. Where the defection is the strongest?

(c) [**5 points**] Integrate Eq. (5) in time and calculate the magnitude of horizontal displacement s by the time when the object hits the ground.

Use the method of successive approximations to solve Eq. (5). In zeroth approximation, neglect the Coriolis term and calculate the velocity $\boldsymbol{v}_0(t)$ in this approximation. Then substitute $\boldsymbol{v}_0(t)$ in the right-hand side of Eq. (5) for $\boldsymbol{v}(t)$ and integrate in time to find the first-order correction $\boldsymbol{v}_1(t)$ due to the Coriolis term and the corresponding horizontal displacement s.

Show that the relative displacement $s/h = C(t/T)$ is proportional to the ratio of the fall time t to the period of the Earth rotation T and calculate the numerical coefficient C. For realistic free-fall experiments, t/T is a small parameter, which justifies the method of successive approximations.

(d) [**2 points**] Determine the numerical value of the displacement s for $h = 100$ m and co-latitude $\theta = 45°$.

The horizontal displacement was actually measured experimentally by dropping objects into deep mines in the 19th century.