Homework #3 — PHYS 374 — Fall 2006 Professor Victor Yakovenko
Deadline: **Tuesday, October 3, 2006, in class** Office: 2115 Physics

Textbook: Roel Snieder, *A Guided Tour of Mathematical Methods for the Physical Sciences*, Cambridge University Press, 2nd edition, 2004, ISBN 0-521-83492-9

Web page: http://www2.physics.umd.edu/˜yakovenk/teaching/

Do not forget to write your name and the homework number!

Ch. 4 Spherical and Cylindrical Coordinates

1. **Volume of a sphere [2 points]**

   Problem 4.4.h from the textbook. (Use the volume element dV given in Problem 4.4.d.)

2. **Derivatives $\partial\hat{\boldsymbol{r}}/\partial\theta$, etc. [3 points]**

   Problem 4.1.h from the textbook. You may formally differentiate (4.7) or use geometrical reasoning to derive (4.12).

3. **Velocity $\boldsymbol{v}$ in spherical coordinates [6 points]**

   Problems 4.3.a,b,c from the textbook. Derive (4.23) and (4.24). Give geometrical interpretation to the terms in (4.24).

4. **Acceleration $\boldsymbol{a}$ in spherical coordinates [4 points]**

   Problems 4.3.d,e from the textbook. Derive (4.27).

5. **Motion in a spherically-symmetrical potential**

   Let us consider motion of a particle subject to a central force, i.e. a force $\boldsymbol{F}(r)$ that has only the radial component $F_r = F$ and depends only on the distance r from the center. The typical examples are Coulomb's electrostatic force and Newton's gravitational force.

   (a) [**2 points**] The angular momentum is defined as $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m\,\boldsymbol{r} \times \dot{\boldsymbol{r}}$. Show that $d\boldsymbol{L}/dt = m\,\boldsymbol{r} \times \boldsymbol{a}$. Show that, for a radial force, $a_\theta = a_\varphi = 0$, so $d\boldsymbol{L}/dt = 0$, and the angular momentum $\boldsymbol{L}$ is conserved.

   (b) [**3 points**] Express the components of the vector $\boldsymbol{L} = (L_r, L_\theta, L_\varphi)$ in the spherical coordinates and the square $L^2 = (\boldsymbol{L} \cdot \boldsymbol{L})$ in terms of the velocity components (v_θ, v_φ).

   (c) [**3 points**] For a radial force, $a_\theta = a_\varphi = 0$, so the second and third equations in (4.27) are equal to zero. Multiply the second equation by v_θ and the third equation by v_φ and add the equations. Solve the resulting equation for $v_\theta^2 + v_\varphi^2$ vs. r. What is the physical meaning of the solution?

(d) [**3 points**] In the first equation (4.27), express $v_\theta^2 + v_\varphi^2$ in terms of the conserved variable L^2. Write the radial equation of motion in the form

$$m\frac{dv_r}{dt} = F(r) + \frac{L^2}{mr^3} \tag{1}$$

Multiply Eq. (1) by $v_r = dr/dt$ and integrate over time. Obtain conservation of energy E in the form

$$E = \frac{m\dot{r}^2}{2} + V_{\text{eff}}(r), \qquad V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2}, \tag{2}$$

where $V(r)$ is the potential energy corresponding to $F = -dV/dr$. The effective potential energy $V_{\text{eff}}(r)$ for radial motion is the sum of $V(r)$ and the centrifugal energy. Show that the latter is nothing but the kinetic energy of angular motion:

$$\frac{L^2}{2mr^2} = m\,\frac{v_\theta^2 + v_\varphi^2}{2}, \tag{3}$$

(e) [**3 points**] In a spherically-symmetrical potential, the particle moves in one plane perpendicular to the vector $\boldsymbol{L}$. Let us select the z axis along $\boldsymbol{L}$, so that the plane of motion corresponds to $\theta = \pi/2$. Because $\theta =$ const, the particle has only two degrees of freedom: r and φ. Show that conservation of angular momentum can be written as

$$mrv_\varphi = mr^2\dot{\varphi} = L. \tag{4}$$

Show that the third equation (4.27) with $a_\varphi = 0$ and $v_\theta = 0$ reproduces Eq. (4).

(f) [**2 points**] Rewrite the conservation laws (2) and (4) as

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}[E - V_{\text{eff}}(r)]}, \qquad \frac{d\varphi}{dt} = \frac{L}{mr^2}. \tag{5}$$

Show that the radial motion $r(t)$ can be calculated from the first equation (5) as

$$\int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}[E - V(r) - \frac{L^2}{2mr^2}]}} = \int_{t_0}^{t} dt. \tag{6}$$

Show that the trajectory $r(\varphi)$ can be obtained as

$$\int_{r_0}^{r} \frac{\frac{L}{mr^2}\,dr}{\sqrt{\frac{2}{m}[E - V(r) - \frac{L^2}{2mr^2}]}} = \int_{\varphi_0}^{\varphi} d\varphi. \tag{7}$$

6. Motion of the Coulomb potential

Now let us apply the general results of Problem 5 for the attractive potential $V = -\alpha/r$, which corresponds to the electrostatic potential for $\alpha = e^2$ and to the gravitational potential for $\alpha = GmM$.

(a) [**2 points**] In this case, sketch the effective potential $V_{\text{eff}}(r)$ (2) for $L \neq 0$. What is the significance of the minimum on the curve $V_{\text{eff}}(r)$.

(b) [**3 points**] For a given $L \neq 0$, what is the minimal energy $E_{\min}$ that the particle can have? What is the shape of its trajectory for $E = E_{\min}$? What is the radius of this trajectory? *Hint: Calculate the radius first and then* $E_{\min}$*?*

Apply these results to the hydrogen atom by substituting $\alpha \to e^2$ and $L \to \hbar$. Show that this gives the Bohr radius and energy, up to numerical factors.

(c) [**3 points**] For given $L \neq 0$ and $0 > E > E_{\min}$, describe qualitatively the time dependence $r(t)$ and the trajectory $r(\varphi)$. Calculate the minimal and maximal distances $r_{\min}$ and $r_{\max}$.

Those students who like mathematical challenges, may attempt to take the integral (7) for $V = -\alpha/r$ *and show that the resulting trajectory* $r(\varphi)$ *is an ellipse. However, this part is not required.*

7. Additional conservation law for the Coulomb potential [**6 points**]

For the potential $V = \alpha/r$ (any sign of α), show that the following vector $\boldsymbol{C}$ is conserved

$$\boldsymbol{C} = \boldsymbol{v} \times \boldsymbol{L} + \frac{\alpha \boldsymbol{r}}{r}. \tag{8}$$

To prove this statement, take the time derivative $d\boldsymbol{C}/dt$ and use Newton's equations of motion to show that $d\boldsymbol{C}/dt = 0$. To handle the double vector product, you can use the identity

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\, \boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\, \boldsymbol{C}. \tag{9}$$

What is the magnitude and the direction of $\boldsymbol{C}$ for a circular trajectory? The direction of $\boldsymbol{C}$ for an elliptic trajectory? Answer qualitatively using symmetry reasoning.

The existence of the additional conservation law does not follow from any general principles (symmetries) and reflects a hidden symmetry unique to the Coulomb potential potential α/r*. This additional conservation law is responsible for the degeneracy of the energy levels of the hydrogen atom in quantum mechanics.*