Homework #2 — PHYS 374 — Fall 2006 Professor Victor Yakovenko
Deadline: **Tuesday, September 26, 2006, in class** Office: 2115 Physics

Textbook: Roel Snieder, *A Guided Tour of Mathematical Methods for the Physical Sciences*, Cambridge University Press, 2nd edition, 2004, ISBN 0-521-83492-9

Web page: http://www2.physics.umd.edu/˜yakovenk/teaching/

Do not forget to write your name and the homework number!

Ch. 3 Power Series

1. **Relativistic energy [9 points]**

The total energy of a free particle of mass m and speed v in special relativity is

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \tag{1}$$

where c is the speed of light.

(i) Write out the Taylor expansion of Eq. (1) in powers of v/c up to $(v/c)^4$. (ii) Identify the physical meaning of the terms in the series. (iii) For an electron in the ground state of a hydrogen atom $v/c \sim 0.01$. What order of magnitude is the relativistic correction to the ground state energy compared with the non-relativistic value?

2. **Speed of surface waves on shallow or deep water**

In HW#1 you used dimensional analysis to find the speed of surface waves on water. When surface tension is irrelevant, and gravity is the only restoring force, these are called "gravity waves". The result was $v \propto \sqrt{gh}$ for shallow water ($h \ll \lambda$) and $v \propto \sqrt{g\lambda}$ for deep water ($\lambda \ll h$). It turns out that the formula interpolating between these two limits is

$$v_{ph} = \sqrt{\frac{g \tanh(kh)}{k}}, \tag{2}$$

where $k = 2\pi/\lambda$ is the wave vector. Here $\tanh(x)$ is the hyperbolic tangent defined as

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh(x)}{\cosh(x)}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}, \tag{3}$$

where $\sinh(x)$ and $\cosh(x)$ are the hyperbolic sine and cosine.

(a) [**3 points**] Obtain the Taylor expansion of $\tanh(x)$ for small $x \ll 1$ up to the order of x^3. You can use expansions (3.14) and (3.15) from the textbook, or take derivatives of $\tanh(x)$ straightforwardly for Eq. (3.11).

(b) [**3 points**] For $x \gg 1$, show that $\tanh(x) \approx 1 - 2e^{-2x}$ in the leading and next-to-leading order.

(c) [**3 points**] Using the approximations for $\tanh(x)$ derived in parts 2a and 2b, show that Eq. (2) reduces to the forms of v quoted above in the shallow $kh \ll 1$ and deep $kh \gg 1$ limits in the leading order.

(d) [**3 points**] Using the approximations for $\tanh(x)$ derived in parts 2a and 2b in the next-to-leading order, derive corrections to the forms of v quoted above in the shallow and deep limits.

3. Lagrange points L1 and L2

If the Earth's orbit around the Sun were perfectly circular, there would be a point L1 inside the Earth's orbit where a satellite (of negligible mass) would orbit the Sun with precisely the same angular velocity as the Earth, since the extra pull of the Sun would be canceled by the pull of the Earth. (The gravitational pull of all other bodies on the Sun is neglected.) There would be a similar point L2 just outside the Earth's orbit, where the smaller pull of the Sun would be supplemented by the pull of the Earth. These are called "Lagrange points".

(a) [**3 points**] Write an equation expressing the condition that the orbital frequency at L1 is equal to that of the Earth, and similarly one for L2. Your equations should be based on Newton's law of gravity and might involve Newton's constant G, the masses of the Sun and the Earth m_s and m_e, the distance d_s from the Sun to the Earth, and the distance $d_{1,2}$ from the Earth to the Lagrange points. (*Hint*: recall that centripetal acceleration on a circle can be written as $r\omega^2$.)

(b) [**1 points**] Re-write your answer to the previous part using only the dimensionless ratios $\mu = m_e/m_s$ and $\delta_{1,2} = d_{1,2}/d_s$.

(c) [**3 points**] Find the solutions for $\delta_{1,2}$ to leading order in μ. (They should be identical.) One approach is to first express the equation as a fifth order polynomial equation, and then ascertain which terms give the leading order solution.

(d) [**2 points**] Put in the mass ratio $\mu = 1/332,830$ and find $\delta_{1,2}$, then put in the (mean) solar distance $d_s \simeq 1.5 \times 10^8$ km and find $d_{1,2}$ in km.

(e) [**3 points**] Show that the next-to-leading order term in $\delta(\mu)$ is $-(1/3)(\mu/3)^{2/3}$. You can do this by writing $\delta = \delta_0 + \epsilon$, where δ_0 is the leading order term you found in part 3c, and ϵ is the next-to-leading order term, and once again ascertaining which terms give the leading order solution for ϵ. How large is this correction compared to the leading order term?

4. The total distance traveled by a bouncing ball [3 points]

Problem 3.3f from the textbook.

5. Ch. 3.4 Transmission through multiple interfaces [10 points]

Let us interpret the transmission and reflection coefficients T and R discussed in Ch. 3.4 of the textbook as the *probabilities* of transmission and reflection. Then, according to the laws of probability, $T + R = 1$, because the incident particle or wave is either transmitted or reflected by the interface.

(a) Using the relation $R = 1 - T$, rewrite Eq. (3.42) from the textbook in term of T only, eliminating R.

(b) Rewrite this equation as a recursive relation between $1/T_{n+1}$ and $1/T_n$.

(c) This recursive relation is rather simple. Solve it and obtain T_n explicitly in terms of the parameters $(1 - t_j)/t_j$ characterizing each interface.

(d) If all interfaces are the same, so that $(1 - t_j)/t_j = (1 - t)/t$ for all j, find how T_n depends on the number of interfaces n.

(e) One might naively guess, ignoring multiple reflections, that $T_n \sim t^n$, because the particle or wave has to pass through n interfaces. Explain qualitatively why this naive (incorrect) dependence on n differs from the exact result that you obtained.

6. Magnetic susceptibility and ferromagnetic phase transition

When an magnetic atom (such as iron) is placed in a weak (infinitesimal) magnetic field B_i, it develops a magnetic moment m_i proportional to B_i:

$$m_i = \chi_0 B_i, \qquad \chi_0 = \frac{C}{T}, \tag{4}$$

where χ_0 is the magnetic susceptibility of the atom. The label i is the atom number in a crystal consisting of many atoms. The magnetic susceptibility χ_0 is inversely proportional to the temperature T (the Curie law), and the coefficient C is called the Curie constant.

Magnetic moments of the atoms produce an additional magnetic field besides the external applied field B_{ext}. The total magnetic field experienced by an atom i depends on magnetic moments m_j of other atoms

$$B_i = B_{\text{ext}} + \sum_j J_{i-j}\, m_j, \tag{5}$$

where J_{i-j} are some coefficients that depends on the crystal structure.

(a) [**2 points**] Because all atoms are the same, one can use the so-called mean-field approximation, and set $m_i = m$ and $B_i = B$. Eliminating B from Eqs. (4) and (5), obtain a relation between m and B_{ext} in the form

$$m_i = \chi B_{\text{ext}}, \tag{6}$$

where χ is the full (renormalized) magnetic susceptibility of the ensemble of atoms. Derive a formula for χ in terms of χ_0 and $g = \sum_k J_k$.

(b) [**2 points**] Discuss $\chi(T)$ as a function of temperature T. Is there a particular temperature T_c where $\chi \to \infty$. Obtain a formula for T_c. What physical phenomenon happens $T = T_c$ and $T < T_c$?

(c) [**2 points**] Let us treat the interaction g between atoms as a small parameter and develop a perturbation theory (an expansion) in powers of g. Rewrite your result for χ as a power series in this parameter. What can you say about convergence of this series when $T \to T_c$?