Homework #10 — PHYS 374 — Fall 2006 Professor Victor Yakovenko
Deadline: **Tuesday, December 5, 2006, in class** Office: 2115 Physics

Textbook: Roel Snieder, *A Guided Tour of Mathematical Methods for the Physical Sciences*, Cambridge University Press, 2nd edition, 2004, ISBN 0-521-83492-9

Web page: http://www2.physics.umd.edu/˜yakovenk/teaching/

Do not forget to write your name and the homework number!

Vector Calculus and Conservation Laws, Ch. 5–12

1. **Plasma oscillations** [**8 points**]

   Consider an electically charged fluid, such as electrons in metals (or ions in plasma). It is desribed by the following set of equations

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = \frac{e}{m_e}\boldsymbol{E}, \qquad \text{equation of motion} \tag{1}$$

$$\frac{\partial \rho_e}{\partial t} + \boldsymbol{\nabla}\cdot(\rho_e\boldsymbol{v}) = 0, \qquad \text{continuity equation} \tag{2}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{\rho_e - \rho_0}{\epsilon_0}, \qquad \text{Gauss's law} \tag{3}$$

   Here e amd m_e are the electron charge and mass, $\boldsymbol{E}$ is the electric field, ρ_e and $\boldsymbol{v}$ are the electic charge density and velocity of the fluid, and ρ_0 is the uniform density of neutralizing charge (produced by the ion lattice in metals).

   (a) [**3 points**] Consider small deviations $\delta\rho_e = \rho_e - \rho_0$ of the electron change density, which produce a small electric field $\boldsymbol{E}$ and motion with small velocity $\boldsymbol{v}$. Linearize Eqs. (1), (2), and (3) to the first order in these small variablses.

   (b) [**3 points**] By eliminating $\boldsymbol{E}$ and $\boldsymbol{v}$ from the linearized equations, obtain the following equation for the electric charge density

$$\frac{\partial^2}{\partial t^2}\,\delta\rho_e = \frac{e\rho_0}{m_e\epsilon_0}\,\delta\rho_e \tag{4}$$

   (c) [**2 points**] Show that $\delta\rho_e(\boldsymbol{r},t)$ oscillates in time with the so-called plasma frequency

$$\omega_p^2 = \frac{ne^2}{m_e\epsilon_0}, \tag{5}$$

   where n is the number density of electrons (the number of electrons per unit volume).

2. **Hydrodynamic formulation of quantum mechanics** [**17 points**]

   Problems from the textbook 8.4 a [**1 pt**], b [**1 pt**], c [**2 pts**], e [**1 pt**].

   Problems from the textbook 11.7 a [**3 pts**], b [**1 pt**], c [**1 pt**], d [**1 pt**], e [**2 pts**], f [**2 pts**], g [**2 pts**].

3. Solving the heat equation in N dimensions [8 points]

Consider the heat equation (11.32) (also known as the diffusion equation) in N dimensions

$$\frac{\partial T}{\partial t} = \kappa \boldsymbol{\nabla}^2 T = \kappa \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \ldots \right) T. \tag{6}$$

for the field $T(\boldsymbol{r}, t)$.

Show that, for a given initial configuration $T_0(\boldsymbol{r}) = T(\boldsymbol{r}, t = 0)$, the field $T(\boldsymbol{r}, t)$ at the subsequent times t is given by the following convolution integral

$$T(\boldsymbol{r}, t) = \int G(\boldsymbol{r} - \boldsymbol{r}', t)\, T_0(\boldsymbol{r}')\, d^N r'. \tag{7}$$

Calculate the kernel $G(\boldsymbol{r} - \boldsymbol{r}', t)$ (which is called Green's function).

Suggestions for the derivation. Fourier-transform Eq. (6) and the field $T(\boldsymbol{r}, t)$ to $\tilde{T}(\boldsymbol{k}, \omega)$. The Fourier-transformed Eq. (6) gives a relation $\omega(\boldsymbol{k})$ between ω and $\boldsymbol{k}$. Fourier-transform $\tilde{T}(\boldsymbol{k}, \omega(\boldsymbol{k}))$ back to $T(\boldsymbol{r}, t)$ and obtain the relation (7). Ch. 19.1 may be useful, although it contains material that we have not discussed.

4. Energy density and energy flux for the electromagnetic field [4 points]

Maxwell's equations in differential form in the Gaussian system of units have the following form:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho \qquad \text{Gauss's law (electric)} \tag{8}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad \text{Gauss's law (magnetic)} \tag{9}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{Faraday's law} \tag{10}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \qquad \textit{Ampère-Maxwell's law} \tag{11}$$

where c is the speed of light, ρ is the electric charge density, and $\boldsymbol{j}$ is the electric current density.

The energy density u of the electromagnetic field (see Problem 4d of HW 1) and the energy flux $\boldsymbol{S}$ (the Poynting vector) are

$$u = \frac{1}{8\pi}(\boldsymbol{E}^2 + \boldsymbol{B}^2), \qquad \boldsymbol{S} = \frac{c}{4\pi}\, \boldsymbol{E} \times \boldsymbol{B}. \tag{12}$$

Show that the continuity equation for the electromagnetic energy is

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{S} = -\boldsymbol{E} \cdot \boldsymbol{j}. \tag{13}$$

The last term in Eq. (13) accounts for the work done by the electric field on motion of charges. Typically, it represents dissipation of electromagentic energy (conversion into Joule's thermal energy of a resistor). Indeed, according to Ohm's law, $\boldsymbol{j} = \sigma \boldsymbol{E}$ (where σ is electric conductivity), so the left-hand-side term in Eq. (13) is negative $-\sigma \boldsymbol{E}^2$.

Suggestions for the proof. Differentiate u in time in Eq. (12) and eliminate the time derivatives of $\boldsymbol{E}$ and $\boldsymbol{B}$ using Maxwell's equations (10) and (11).

5. Fun with curls [17 points]

(a) [**3 points**] For arbitrary vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, show that

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\,\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\,\boldsymbol{C}. \tag{14}$$

If the vectors involve differential operation, then we cannot change their order, so Eq. (14) should be written in the component notation as

$$[\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})]_i = A_j B_i C_j - A_j B_j C_i, \tag{15}$$

where summation over the repeated indices j is implied.

(b) [**2 points**] Using Eq. (15), show that

$$\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) = \frac{1}{2}\,\boldsymbol{\nabla}v^2 - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}. \tag{16}$$

According to Eq. (7.5) in the textbook, $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v}$ is called vorticity. So, Eq. (16) can be also written as

$$(\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} = \frac{1}{2}\,\boldsymbol{\nabla}v^2 - \boldsymbol{v}\times\boldsymbol{\omega}. \tag{17}$$

(c) [**2 points**] Show that, for any scalar function f and any vector function $\boldsymbol{v}$, the following identities hold

$$\mathrm{curl}\,\mathrm{grad}\,f = \boldsymbol{\nabla}\times\boldsymbol{\nabla}f = 0, \qquad \mathrm{div}\,\mathrm{curl}\,\boldsymbol{v} = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v}) = 0 \tag{18}$$

(d) [**4 points**] Let us consider the equation of motion for an ideal fluid (no viscosity) in a gravitational potential Φ:

$$\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p - \boldsymbol{\nabla}\Phi. \tag{19}$$

First, let us consider the case of incompressible fluid, where $\rho = \mathrm{const}$. Taking curl of Eq. (19), **show** that the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v}$ satisfies the following equation

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}). \tag{20}$$

Second, **show** that Eq. (20) is also valid for a barotropic fluid, where the density $\rho(p)$ is a function of pressure p.

Eq. (20) shows that, if initially $\boldsymbol{\omega} = 0$, the ideal fluid will stay irrotational with $\boldsymbol{\omega} = 0$.

Show that for the quantum fluid in Eq. (11.59), vorticity $\boldsymbol{\omega}$ is zero (unless it has quantized vortices).

(e) [**2 points**] Using Eq. (15), show for an arbitrary vector function $\boldsymbol{A}$ that

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}\,(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \boldsymbol{\nabla}^2\,\boldsymbol{A}. \tag{21}$$

(f) [**4 points**] Let us consider Maxwell's equation in vacuum, where $\rho = 0$ and $\boldsymbol{j} = 0$. Taking curl of Eq. (10), using Eq. (21) and Eqs. (8) and (11), show that the electric field satisfis the wave equation. Show the same for the magnetic field as well:

$$\boldsymbol{\nabla}^2\,\boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} = 0, \qquad \boldsymbol{\nabla}^2\,\boldsymbol{B} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{B}}{\partial t^2} = 0. \tag{22}$$