Homework #1 — PHYS 374 — Fall 2006 Professor Victor Yakovenko
Deadline: **Tuesday, September 19, 2006, in class** Office: 2115 Physics

Textbook: Roel Snieder, *A Guided Tour of Mathematical Methods for the Physical Sciences*, Cambridge University Press, 2nd edition, 2004, ISBN 0-521-83492-9

Web page: http://www2.physics.umd.edu/~yakovenk/teaching/

Do not forget to write your name and the homework number!

Ch. 2 Dimensional Analysis

1. **Flow rate through a pipe**

   (a) [**8 points**] Problems 2.6 a,b,c from textbook. Also, derive an expression for the fluid velocity at the center of the tube using dimensional analysis. Discuss velocity before answering question c.

   (b) [**2 points**] Suppose, because of cholesterol deposits, the radius of somebody's blood vessels has decreased by a factor of two: $R \to R/2$.

   Assuming that the heart maintains the same blood pressure, by what factor the blood flow (and, thus, the oxygen flow) would decrease?

   Suppose the heart tries to maintain the same blood and oxygen flow by increasing blood pressure (and, thus, the pressure gradient across the blood system). By what factor the pressure has to be increased?

   Do you think the person could survive these conditions?

2. **Speed of surface waves on water**

   The *phase velocity* of a wave is the speed of wavefronts of constant phase, while the group velocity is the speed of a wavepacket. Both have dimensions of velocity, of course, so dimensional analysis will not distinguish them.

   The speed of water waves might depend upon various quantities: (i) properties of the water such as density ρ or surface tension σ (energy per unit area of stretching of a surface), (ii) properties of the environment such as gravitational acceleration g or depth of the body of water h, and (iii) the wavelength of the wave λ. Depending on the wavelength of the wave different quantities are important in determining the wave speed. The speed is determined by the restoring force(s) that tend to return the surface to equilibrium, and the inertia of the water.

   (a) [**3 points**] First consider wavelengths long enough that surface tension can be neglected but short enough that the depth of the body of water can be neglected (*deep water waves*). (a) Use dimensional analysis to find how the wave speed depends on the remaining quantities ρ, g, λ. (b) Estimate (modulo the unknown dimensionless coefficient) the speed of a one meter wave on a lake.

(b) [**3 points**] Next consider wavelengths much longer than the depth (*shallow water waves*). Now the speed may depend upon the depth h as well, so there is a dimensionless ratio h/λ that the speed could depend on. However, one might reason that when this ratio is very small, the dependence on λ should drop out, since the restoring force is a local process and the concept of wavelength involves very distant parts of the wave. How then would the wave speed depend upon the various quantities ρ, g, h? Estimate (modulo the unknown dimensionless coefficient) the speed of a (long wavelength) tsunami on the Pacific ocean of depth 4 km.

(c) [**3 points**] Finally consider wavelengths short enough that the surface tension is the dominant restoring force and gravity can be neglected (*capillary waves*). How does the speed of capillary waves depend upon the remaining quantities ρ, λ, σ?

(d) [**3 points**] The phase velocity v_{ph} of a wave of angular frequency ω and wave vector k is $v_{ph} = \omega/k$, while the group velocity v_{gr} of a wavepacket consisting of waves with wave vector near a given k is $v_{gr} = d\omega/dk$. What is the exact ratio of group velocity to phase velocity v_{gr}/v_{ph} for the three cases considered above? (Recall that $k = 2\pi/\lambda$.)

(e) [**3 points**] In which of the above cases does the wave speed depend upon the density of the water? Explain physically why this is the case.

3. Planck length

(a) [**3 points**] Using dimensional analysis, construct a fundamental unit of length l_P, called the Planck length, from the fundamental constants of nature:

- Planck's constant $\hbar \approx 1 \times 10^{-34}$ J s
- the speed of light $c \approx 3 \times 10^8$ m/s
- Newton's gravitational constant $G \approx 6.7 \times 10^{-11}$ N m^2/kg^2

Estimate the numerical value of l_P and compare it with the proton radius of order of 1 fm $= 10^{-15}$ m.

(b) [**3 points**] Using dimensional analysis with the same fundamental constants, derive the Planck energy associated with the Planck length.

Estimate the numerical value of the Planck energy in electron-Volts (1 eV $\approx$ 1.6×10^{-19} J). Compare the Planck energy with the highest energy achieved in modern accelerators of elementary particles. (The LHS collider is expected to start operations in November 2007 and achieve 14 TeV.) Is it realistic to reach the Planck energy in accelerators in foreseeable future?

4. Gaussian electromagnetic units

The SI system has an independent unit of charge, the Coulomb. The electric and magnetic fields produced by a point charge q_1, and the force acting on a charge q_2 are

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{\boldsymbol{r}}, \quad \boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{q_1\boldsymbol{v}_1 \times \hat{\boldsymbol{r}}}{r^2}, \quad \boldsymbol{F} = q_2\boldsymbol{E} + q_2\boldsymbol{v}_2 \times \boldsymbol{B}, \quad c = \frac{1}{\sqrt{\epsilon_0\mu_0}}, \qquad (1)$$

where ϵ_0 and μ_0 are dimensional constants, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the velocities of the charges, and $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ is the unit vector (dimensionless). The speed of light c is expressed in terms of ϵ_0 and μ_0.

While the SI system is widely used in electrical engineering, a more natural system of units, called the Gaussian system, is used in theoretical physics. In the Gaussian system, the Coulomb law has the simple form:

$$\boldsymbol{F} = \frac{q_1 q_2}{r^2}\,\hat{\boldsymbol{r}} \tag{2}$$

without any dimensional constant, and the unit of charge is derived from this law.

(a) [**3 points**] Using Eq. (2), express the unit of electron charge in the Gaussian system in terms of the units of Length, Mass, and Time.

Traditionally, the Gaussian system uses the CGS units (centimeter, gram, second), rather than MKS units (meter, kilogram, second). In the CGS Gaussian system, the electron charge e has the value 4.8×10^{-10} e.s.u. (electrostatic unit of charge).

(b) [**3 points**] In the Gaussian system, Eq. (1) becomes

$$\boldsymbol{E} = \frac{q_1}{r^2}\,\hat{\boldsymbol{r}}, \quad \boldsymbol{B} = q_1\frac{\boldsymbol{v}_1}{c}\times\frac{\hat{\boldsymbol{r}}}{r^2}, \quad \boldsymbol{F} = q_2\boldsymbol{E} + q_2\frac{\boldsymbol{v}_2}{c}\times\boldsymbol{B}. \tag{3}$$

In these equations, ϵ_0 has been eliminated by redefining the electric charge in the Coulomb law: $q_1q_2/4\pi\epsilon_0 \to q_1q_2$, and μ_0 has been eliminated by using the speed of light: $\mu_0 \to 1/c^2\epsilon_0$.

Calculate the (dimensionless) ratio of the magnetic and electric forces acting between two point charges using Eq. (1) and Eq. (3) and show that they give the same result. For non-relativistic velocities, which force is greater, magnetic or electric, and by what factor?

(c) [**3 points**] In the Gaussian system, derive the units of $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of Length, Mass, and Time.

Using dimensional analysis, derive an expression for the energy density u of the electric field $\boldsymbol{E}$ in the Gaussian system. The dimensionality of u is Energy/Volume. You may argue that it can depend on $\boldsymbol{E}$ and, perhaps, on the fundamental constant c. Also derive the energy density of the magnetic field $\boldsymbol{B}$.

(d) [**3 points**] In the SI system, the energy density of the electric and magnetic fields is

$$u = \frac{\epsilon_0\boldsymbol{E}^2}{2} + \frac{\boldsymbol{B}^2}{2\mu_0}. \tag{4}$$

From Eq. (4), derive an exact expression for the energy density u in the Gaussian system of units.

5. Bohr length and Bohr energy

(a) [**3 points**] Using dimensional analysis, derive a formula (in the Gaussian system of units) for the radius of the hydrogen atom, called the Bohr radius a_B. You may argue that it can depend on the electron charge e, the electron mass m, and the Planck constant $\hbar$. Estimate the numerical value of a_B in Angstrom (10^{-10} m).

(b) [**3 points**] Using dimensional analysis, derive a formula (in the Gaussian system of units) for the energy of the hydrogen atom. Estimate its numerical value in eV. Using dimensional analysis, estimate electric voltage produced by a typical battery.

6. Fine-structure constant

[**3 points**] Show that it is possible to make a dimensionless combination $\alpha = e^2/\hbar c$ of the fundamental constants e, c, and $\hbar$. (This formula is written in the Gaussian system.) Calculate the numerical value of α. The parameter α, called the fine-structure constant, serves as the expansion parameter in quantum electrodynamics, which combines electrodynamics (e and c) and quantum mechanics ($\hbar$).